Introduction to Quantum Field Theory

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Version: May 2, 2011

Foreword

These lecture notes are meant to give a concise introduction to Quantum Field Theory using path integral methods. They are mainly based on the books by W. Cottingham and D. Greenwood [1] (Chapter 1), A. Das [2] (Chapter 2) and L. Ryder [3] (Chapter 3) with supplemental material from Refs. [4–6]. The interested reader may want to consult these books for further details.

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1. Introduction

1.1. A brief history of Quantum Field Theory

Quantum electrodynamics was formulated about 1950, many years after Planck's original hypothesis (1901) that the electromagnetic field should be quantized. In subsequent years, quantum field theories were studied extensively and one was finally lead to the "Standard Model" of particle physics which describes elementary particles and their interactions in flat space-time. Three such basic interactions are involved:

1. the electromagnetic field, whose gauge bosons are massless spin 1 "photons",
2. the weak field, whose gauge bosons are the massive spin 1 $W^{\pm}$, Z bosons,
3. and the strong field (QCD), whose gauge bosons are the (postulated) 8 massless spin 1 "gluons".

The first two forces are described by the Weinberg-Salam electroweak model based on the gauge group $U(1) \times SU(2)$, and the $W^{\pm}$, Z bosons become massive due to spontaneous symmetry breaking through the so-called Higgs mechanism. The strong force, on the other hand is based on the gauge group $SU(3)$ whose three (anti)charges are dubbed (anti)red, (anti)green and (anti)blue, hence the term "chromo" (colour) in the models name.

Additionally, the elementary particles charged under these forces can be categorized into three generations of quark-doublets and equally many generations of lepton doublets. Details are given in the Table 1.1

Table 1.1.: The elementary particle zoo

	1. generation		2. gen.		3. gen.	
quark	up (u)	down (d)	charm (c)	strange (s)	top (t)	bottom (b)
charge	$+\frac{2}{3}$	$-\frac{1}{3}$	$+\frac{2}{3}$	$-\frac{1}{3}$	$+\frac{2}{3}$	$-\frac{1}{3}$
mass	1.5-4MeV	4-8MeV	1.15-1.35GeV	80-130MeV	169-174GeV	4.1-4.4GeV
lepton	electron e^-	el.-neutrino ν_e	muon μ^-	μ-neutr. ν_μ	tauon τ^-	τ-neutr. ν_τ
charge	-1	0	-1	0	-1	0
mass	511keV	< 3eV	105.66MeV	< 100eV	1.777GeV	< 30MeV

Many experiments, mostly particle accelerators, have verified the validity of the standard model. The only unobserved particle to date is the Higgs particle for which the search is ongoing. Furthermore, the standard model has known shortcomings: In its original form, massless neutrinos are considered which meanwhile is known to be untrue. Neutrinos do in fact have mass and according extensions to the standard model have been suggested. Furthermore, one expects that at energies at/or beyond the TeV scale new physical effects which are not covered by the standard model will appear.

Some important discoveries were:

- the electron e^- 1897 by J.J. Thomson,
- the anti-electron (or positron e^+) 1932 by C.D. Anderson in cosmic radiation,
- and in the same year J. Chadwick discovered the neutron,
- the muon μ^- 1936 by C.D. Anderson in cosmic rays,
- the $W^\pm, Z$ bosons (80/90GeV) in 1983 at CERN,

just to name a few.

History and overview of some particle accelerators.

1. Early 1930s: Cockroft-Walton linear accelerator at Cambridge, UK (0.7 MeV); and Lawrence's cyclotron at Berkley, USA (1.2 MeV);
2. further accelerators in the 1950s and after;
3. TEVATRON (Fermilab), p-$\bar{p}$ collisions, 900 GeV, 1987-2011, discovered top-quark;
4. SLC (SLAC, Stanford), e^--e^+ collisions, 50 GeV, 1989-1998;
5. HERA (DESY, Hamburg) e at 30 GeV, p at 820 GeV, 1992-2007;
6. LEP2 (CERN), e^-, e^+, 81 GeV, 1996-2000;
7. Vienna Environmental Research Accelerator (VERA), 3 MV tandem accelerator, for Accelerator Mass Spectrometry (AMS), 1996-present;
8. PEP-II (SLAC, Stanford), e^-, e^+, 9 GeV, 1999-2008,
9. Relativistic Heavy Ion Collider "RHIC" (Brookhaven National Laboratory, New York), 2000-present;
10. LHC (CERN), p, 7 TeV, 2009-present.

Concepts of renormalization/regularization.

Charged particles receive a "self-interaction", i.e. they feel their own field. "Naked charges" can never be measured, only the effective charge including self-interaction of the particle. In fact, the value is energy dependent (RG-flow) as described by the so-called β-function. For example, at low energies the fine-structure constant of QED is given by

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}\,, \tag{1.1}$$

but at high energies α becomes larger, as can be measured in particle accelerators. This can be interpreted by the following gedanken-experiment: vacuum fluctuations constantly lead to the production and annihilation of (virtual) particles and antiparticles. Near a charged particle, say an electron, these will shield some portion of the charge leading e.g. to the value of $1/137$ at low energies. Measuring at high energies means we can detect the charge of that electron at closer distance and less shielding, hence the increased value. In fact, being

interpreted as a point particle, the beta-function diverges as the energy goes to infinity. However, this picture is not expected to hold at arbitrarily high energies.

On the other hand, there are also quantum field theories, where the beta-function behaves in the opposite way: In QCD, for example, the coupling is strong at low energies and becomes weak at high energies. (Such theories are called asymptotically free.) The mathematical reason is that quantumchromodynamics QCD is a non-Abelian gauge theory, meaning its gauge bosons are charged and interact with each other. Therefore, the naive "screening" picture above does not apply here. In fact, so-called "confinement" prevents production of charged particles.

These ideas will be made more explicit in Section 3.

Notation.

Throughout these lecture notes, natural units as described in Appendix A.2 are used.

1.2. Classical theory

1.2.1. From particle mechanics to field theory

Consider a point particle of mass m whose position at time t is $x(t)$, and which is driven by a force $F = V'(x)$. Its equation of motion (i.e. Newton's second law) is then given by

$$m\frac{d^2x}{dt^2} + \frac{dV}{dx} = 0\,. \tag{1.2}$$

A way of deriving this equation is by the principle of least action. The action S in the above case is given by

$$S = \int_{t_1}^{t_2} dt L\,, \qquad\qquad L = \frac{m}{2}\left(\frac{dx}{dt}\right)^2 - V(x)\,, \tag{1.3}$$

where L is called the Lagrangian, and the integral in the action is taken over a path from t_1 to t_2. The principal of least action states that the path the particle actually takes (from the infinitely many possible ones) is the one for which S is a minimum. It can be derived by varying S and requiring this variation to vanish, i.e.

$$\delta S = \int_{t_1}^{t_2} dt \left(m\frac{dx}{dt}\frac{d\delta x}{dt} - \frac{dV}{dx}\delta x\right) = 0\,. \tag{1.4}$$

Integrating the first term by parts keeping in mind that the variation is zero at the end points $t_{1,2}$, we find

$$\int_{t_1}^{t_2} dt \left(m\frac{d^2x}{dt^2} + \frac{dV}{dx}\right)\delta x = 0\,, \tag{1.5}$$

which for arbitrary variations δx implies Eqn. (1.2) above. The above can of course readily be extended to a system of particles, or a more complicated mechanical problem with n degrees

of freedom q_i. The Lagrangian then becomes $L(q_i, \dot{q}_i)$. If we push this generalization further, say to an infinite number of degrees of freedom, we replace the $q_i(t)$ with fields $\phi(x_\mu) = \phi(t,x,y,z)$ where the index i is replaced by continuous variables x, y, z. Furthermore, we define the Lagrangian density $\mathcal{L}$ by $L = \int d^3x \mathcal{L}$, and a general field theory action hence is written as

$$S = \int d^4x \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)) \tag{1.6}$$

depending on fields various fields ϕ_i (which might have additional indices) and their gradients. We furthermore assume "natural boundary conditions", i.e. that these fields vanish sufficiently fast at infinity to justify integration by parts. Variation of this action leads to

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) \delta\phi_i \,, \tag{1.7}$$

implying the generalized Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} = 0 \,. \tag{1.8}$$

1.2.2. Maxwell fields

In the case of photons, the action is given by

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + j_\mu A^\mu \right) , \tag{1.9}$$

where A_μ is the vector potential, j_μ denotes an (external) current density and

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} , \tag{1.10}$$

is the electromagnetic field strength tensor in vacuum. Furthermore, we use the Lorentz metric $\eta_{\mu\nu} = (+,-,-,-)$ for pulling indices up and down, and consider natural units where $c = \hbar = 1$ (cf. Appendix A.2). Varying the action (1.9) with respect to the A_μ leads to the equations of motion[1]

$$\partial_\nu F^{\mu\nu} = \partial_\nu \partial^\nu A^\mu - \partial^\mu (\partial_\nu A^\nu) = -j^\mu \,. \tag{1.11}$$

These are in fact the inhomogeneous Maxwell equations, i.e. for $\mu = 0$ one has with $\vec{E} = -\nabla A^0 - \frac{\partial \vec{A}}{\partial t}$ and $\rho = j^0$:

$$\mathrm{div}\vec{E} = \rho \,, \tag{1.12}$$

whereas using $\vec{B} = \mathrm{rot}\vec{A}$ the $\mu = 1,2,3$ components yield

$$\mathrm{rot}\vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j} \,. \tag{1.13}$$

[1] Note, that these imply current conservation since $0 = \partial_\mu \partial_\nu F^{\mu\nu} = \partial_\mu j^\mu$.

The homogeneous Maxwell equations follow as an identity directly from the definition of the antisymmetric field strength tensor (1.10), i.e.

$$\partial_\nu \widetilde{F}^{\mu\nu} = \partial_\nu \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = \epsilon^{\mu\nu\rho\sigma}\partial_\nu\partial_\rho A_\sigma = 0\,. \tag{1.14}$$

The $\mu = 0$ and $\mu = 1,2,3$ components correspond to

$$\mathrm{div}\vec{B} = 0 \qquad \text{and} \qquad \mathrm{rot}\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0\,, \tag{1.15}$$

respectively.

Exercise 1 *Derive the Maxwell equations starting from* (1.11) *and* (1.14).

The action (1.9) as well as the field strength tensor $F_{\mu\nu}$ are invariant under the "gauge transformation"

$$A'_\mu = A_\mu + \partial_\mu \lambda\,, \tag{1.16}$$

(up to a surface term provided the current density is conserved, $\partial_\mu j^\mu = 0$). This gauge freedom can be fixed by demanding a gauge condition, such as e.g. the Lorenz condition $\partial_\mu A^\mu = 0$.

1.2.3. Scalar field theory

Let us now consider the simplest field theory, which represents a good toy model in order to learn general properties of quantum field theory. We start with a free field whose classical action is given by

$$S_\phi = \int d^4x \left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 - j\phi \right)\,. \tag{1.17}$$

The e.o.m. follow as

$$\left(\Box + m^2\right)\phi(x) = j(x)\,. \tag{1.18}$$

A solution can then be found by employing the Green function method where

$$\begin{aligned} \phi(x) &= \phi^0(x) + \int d^4x G(x,x')j(x')\,, \\ \left(\Box + m^2\right) G(x,x') &= \delta^4(x-x')\,, \end{aligned} \tag{1.19}$$

where ϕ^0 solves the homogeneous equation and is chosen in such a way that ϕ satisfies the boundary conditions. Making use of translational invariance, the first line of (1.19) can be replaced by an algebraic one by making a Fourier transformation leading to

$$G(x-x') = -\int \frac{d^4p}{(2\pi)^4}\frac{e^{\mathrm{i}p(x-x')}}{p^2 - m^2}\,. \tag{1.20}$$

Obviously one has to deal with the zero of the expression $p_0^2 - \vec{p}^2 - m^2$ by prescribing a slightly deformed contour of integration. This in turn leads to the definition of the retarded and advanced Green functions, as follows:

$$\begin{aligned} G_{\substack{\text{ret}\\\text{adv}}}(x-x') &= -\int \frac{d^4p}{(2\pi)^4} \frac{e^{\mathrm{i}p(x-x')}}{2\omega_p} \left(\frac{1}{p_0 - \omega_p \pm \mathrm{i}\varepsilon} - \frac{1}{p_0 + \omega_p \pm \mathrm{i}\varepsilon} \right) \\ &= \frac{\Theta(\pm(x_0 - x_0'))}{(2\pi)^3} \int \frac{d^3p}{\omega_p} e^{\mathrm{i}\vec{p}(\vec{x}-\vec{x}')} \sin\left(\omega_p(x_0 - x_0')\right) , \end{aligned} \tag{1.21}$$

where $\omega_p = \sqrt{\vec{p}^2 + m^2}$.

In order to find the homogeneous solution ϕ^0 we once more employ a Fourier ansatz leading to

$$\begin{aligned} \phi^0(x) &= \int \frac{d^4p}{(2\pi)^3} e^{\mathrm{i}px} \delta(p^2 - m^2) \phi'(p) \\ &= \frac{1}{(2\pi)^3} \int \frac{d^3p}{2\omega_p} \left(\phi^+(\vec{p}) e^{\mathrm{i}px} + \phi^-(\vec{p}) e^{-\mathrm{i}px} \right) , \end{aligned} \tag{1.22}$$

where $\phi^\pm(\vec{p}) := \phi'(\pm\omega_p, \pm\vec{p})$.

1.2.4. Dirac fields

The Schrödinger equation for e.g. an electron wave function ψ is given by

$$\mathrm{i}\frac{\partial \psi}{\partial t} = H\psi \, . \tag{1.23}$$

In order to ensure a symmetry between space and time (as required by special relativity), Dirac postulated the Hamiltonian for a free electron to be of the form

$$H = -\mathrm{i}\vec{\alpha}\vec{\nabla} + \beta m \, , \tag{1.24}$$

so that

$$\left(\mathrm{i}\frac{\partial}{\partial t} + \mathrm{i}\vec{\alpha}\vec{\nabla} - \beta m \right) \psi = 0 \, , \tag{1.25}$$

where α_i and β must be matrices satisfying a Clifford algebra, since the solutions to the above equation should also be solutions to the Klein-Gordon equation leading to

$$\begin{aligned} &\left(\mathrm{i}\frac{\partial}{\partial t} - \mathrm{i}\vec{\alpha}\vec{\nabla} + \beta m \right) \left(\mathrm{i}\frac{\partial}{\partial t} + \mathrm{i}\vec{\alpha}\vec{\nabla} - \beta m \right) \psi \\ &= \left(-\frac{\partial^2}{\partial t^2} + \frac{1}{2} \left\{ \alpha_i, \alpha_j \right\} \partial_i \partial_j + \mathrm{i}m \left\{ \alpha_i, \beta \right\} \partial_i - \beta^2 m^2 \right) \psi \\ &= 0 \, , \end{aligned} \tag{1.26}$$

where

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij} \, , \qquad \beta^2 = 1 \, , \qquad \{\alpha_i, \beta\} = 0 \, . \tag{1.27}$$

Though not unique, a possible representation is the so-called chiral representation where

$$\alpha_i = \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \tag{1.28}$$

and σ^i are the usual Pauli matrices. We may now rewrite Eqn. (1.25) as

$$\left(\mathrm{i}\gamma^\mu \partial_\mu - m\right)\psi = \beta\left(\mathrm{i}\frac{\partial}{\partial t} + \mathrm{i}\vec{\alpha}\vec{\nabla} - \beta m\right)\psi = 0\,, \tag{1.29}$$

where $\gamma^0 = \beta$ and $\gamma^i = \beta\alpha_i$ and $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, $\mu, \nu \in \{0, 1, 2, 3\}$. The Lagrangian must be Hermitian and scalar leading to the above e.o.m., and hence

$$\mathcal{L} = \bar{\psi}\left(\mathrm{i}\gamma^\mu \partial_\mu - m\right)\psi\,, \tag{1.30}$$

where $\bar{\psi} = \psi^\dagger\gamma^0$. Another useful matrix is $\gamma^5 \equiv \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ which in the chiral representation is given by

$$\gamma^5 = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}. \tag{1.31}$$

The matrices $\frac{1}{2}\left(\mathbb{1}_4 \pm \gamma^5\right)$ are projectors giving the right and left handed parts of a Dirac spinor, i.e.

$$\frac{1}{2}\left(\mathbb{1}_4 - \gamma^5\right)\psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}, \qquad \frac{1}{2}\left(\mathbb{1}_4 + \gamma^5\right)\psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix}. \tag{1.32}$$

The Dirac equation describes particles with intrinsic angular momentum $\frac{\hbar}{2}\sigma$ and intrinsic magnetic moment $\frac{q\hbar}{2m}\sigma$ if the particle carries charge q. Furthermore, there exist negative energy solutions which Dirac interpreted as antiparticles. This, of course at the time was a rather bold claim, which however was experimentally confirmed later when the positron was discovered in cosmic radiation in 1932.

1.3. The role of symmetries in physics

Noether's theorem states that every symmetry of the action leads to a conserved quantity. For example, translation symmetry leads to energy-momentum conservation. Derivations of this theorem may be found in many textbooks, and here we only briefly review the following example considering invariance under time-translation: Using the Euler-Lagrange e.o.m. we get

$$\begin{aligned} \frac{dL(q_i, \dot{q}_i)}{dt} &= \frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right). \end{aligned} \tag{1.33}$$

Hence

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right) = 0\,, \tag{1.34}$$

and the energy of the system

$$E = \left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right) \tag{1.35}$$

remains constant during the motion.

Energy-momentum tensor of a scalar field.

Consider a space-time displacement $x^\mu \to x^\mu + \delta a$, where δa is independent from x. By computing the change in the Lagrangian $\mathcal{L}$ of a scalar field ϕ theory corresponding to this displacement, one easily computes the equations expressing conservation of linear momentum and energy: Since $\mathcal{L}$ does not explicitly depend on x, we have

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)\,, \qquad \delta\phi = (\partial_\nu\phi)\delta a^\nu\,. \tag{1.36}$$

Using the fact that $[\delta, \partial_\mu] = 0$ and the Euler-Lagrange equations of motion (1.8), we can rewrite this as

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi\right)\delta a^\nu\,. \tag{1.37}$$

Alternatively, we may derive $\delta\mathcal{L}$ directly as

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial x^\mu}\delta a^\mu\,. \tag{1.38}$$

Since the δa^ν are arbitrary, it hence follows that

$$\partial_\mu T^\mu_\nu = 0\,, \qquad T^\mu_\nu = \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}\right)\,, \tag{1.39}$$

where T^μ_ν is called the *energy-momentum tensor*. The component $T^0_0 = \left(\frac{\partial\mathcal{L}}{\partial(\dot\phi)}\dot\phi - \mathcal{L}\right)$ corresponds to the energy density of the field — cf. Eqn. (1.35). Observe furthermore, that the 0 component of Eqn. (1.39),

$$\frac{\partial}{\partial t}T^0_0 + \vec\nabla\vec{T}_0 = 0\,, \tag{1.40}$$

expresses local conservation of energy, and using the divergence theorem integration over all space leads to

$$\frac{\partial}{\partial t}\int d^3x T^0_0 = 0\,, \tag{1.41}$$

assuming the field vanishes at large distances. Similarly, one derives

$$\frac{\partial}{\partial t}P_i = 0\,, \qquad P_i = \int d^3x T^0_i\,, \tag{1.42}$$

expressing conservation of momentum P_i.

Finally, we should mention, that symmetries which are broken at the quantum level lead to so-called anomalies. (For example, cancellations of anomalies leads to requirement that number of quark and lepton generations match.)

2. Path integral methods

2.1. Path integral formulation in quantum mechanics

We start by recapitulating some basic formulae from quantum mechanics, for simplicity consider a one dimensional system, i.e. with one pair of conjugate operators X and P satisfying $[X, P] = \mathrm{i}$. The eigenstates of the coordinate operator satisfy $X|x\rangle = x|x\rangle$ and define an orthonormal basis, i.e.

$$\langle x \,|\, x' \rangle = \delta(x - x') \,, \qquad\qquad \int dx |x\rangle\langle x| = \mathbb{1} \,. \tag{2.1}$$

Similarly, the eigenstates of the momentum operator satisfy $P|p\rangle = p|p\rangle$ and

$$\langle p \,|\, p' \rangle = \delta(p - p') \,, \qquad\qquad \int dp \, |p\rangle\langle p| = \mathbb{1} \,. \tag{2.2}$$

The transformation operator between the two basis is given by

$$\langle p \,|\, x \rangle = \frac{1}{\sqrt{2\pi}} e^{-\mathrm{i}px} = \langle x \,|\, p \rangle^* \,, \tag{2.3}$$

and hence the Fourier transform of functions is defined as

$$f(x) = \langle x \,|\, f \rangle = \int dp \, \langle x \,|\, p \rangle\langle p \,|\, f \rangle = \frac{1}{\sqrt{2\pi}} \int dp \, e^{\mathrm{i}px} f(p) \,. \tag{2.4}$$

The coordinate states in the Heisenberg picture are related to the Schrödinger states as $|x, t\rangle_H = e^{\mathrm{i}Ht}|x\rangle$. Orthonormality and completeness hold for the Heisenberg states only for equal times, as can be easily checked.

In general, when promoting a classical system to a quantum system, implying that observables are promoted to operators, it is not clear what has to be done with products of non-commuting operators: an ordering prescription, such as 'normal ordering' and 'Weyl ordering' is required. Here we shall consider the latter, also known as 'symmetric ordering'.

We may now calculate the transition amplitude $U(t_f, x_f; t_i, x_i) = {}_H\langle x_f, t_f \mid x_i, t_i \rangle_H$ for $t_f > t_i$ in the Heisenberg picture. Dividing the time interval between the initial and the final time into N equal segments of infinitesimal length ε and introducing complete sets of coordinate basis states for every intermediate time point, we obtain

$$\begin{aligned} U(t_f, x_f; t_i, x_i) = \lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int & dx_1 \cdots dx_{N-1} \, {}_H\langle x_f, t_f \,|\, x_{N-1}, t_{N-1} \rangle_H \\ & \times {}_H\langle x_{N-1}, t_{N-1} \,|\, x_{N-2}, t_{N-2} \rangle_H \cdots {}_H\langle x_1, t_1 \,|\, x_i, t_i \rangle_H \end{aligned} \tag{2.5}$$

with

$${}_H\langle x_k, t_k \,|\, x_{k-1}, t_{k-1} \rangle_H = \langle x_k | e^{-\mathrm{i}(t_k - t_{k-1})H} | x_{k-1} \rangle = \int \frac{dp_k}{2\pi} e^{\mathrm{i}p_k(x_k - x_{k-1}) - \mathrm{i}\varepsilon H\left(\frac{x_k + x_{k-1}}{2}, p_k\right)} , \tag{2.6}$$

using Weyl ordering. Hence, identifying $x_0 = x_i$ and $x_N = x_f$, we arrive at

$$U(t_f, x_f; t_i, x_i) = \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int dx_1 \cdots dx_{N-1} \frac{dp_1 \cdots dp_N}{(2\pi)^N} e^{\mathrm{i}\varepsilon \sum_{k=1}^{N} \left(p_k \left(\frac{x_k - x_{k-1}}{\varepsilon}\right) - H\left(\frac{x_k + x_{k-1}}{2}, p_k\right)\right)}. \tag{2.7}$$

When specializing to the class of Hamiltonians which are quadratic in the momentum variables, e.g. $H(x,p) = \frac{p^2}{2m} + V(x)$, the momentum integrals above become Gaussian, i.e. particularly simple:

$$\int \frac{dp_k}{2\pi} e^{-\mathrm{i}\varepsilon\left(\frac{p_k^2}{2m} - \frac{p(x_k - x_{k-1})}{\varepsilon}\right)} = \sqrt{\frac{m}{2\pi\mathrm{i}\varepsilon}} e^{\frac{\mathrm{i}\varepsilon m}{2}\left(\frac{x_k - x_{k-1}}{\varepsilon}\right)^2}, \tag{2.8}$$

hence leading to Feynman's path integral for the transition amplitude in quantum mechanics

$$\begin{aligned} U(t_f, x_f; t_i, x_i) &= \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \left(\frac{m}{2\pi\mathrm{i}\varepsilon}\right)^{N/2} \int dx_1 \cdots dx_{N-1} e^{\mathrm{i}\varepsilon \sum_{k=1}^{N} \left(\frac{m}{2}\left(\frac{x_k - x_{k-1}}{\varepsilon}\right)^2 - V\left(\frac{x_k + x_{k-1}}{2}\right)\right)} \\ &= \mathcal{N} \int \mathcal{D}x \exp\left(\mathrm{i} \int_{t_i}^{t_f} dt \left(\frac{m}{2}\dot{x}^2 - V(x)\right)\right) \\ &= \mathcal{N} \int \mathcal{D}x\, e^{\mathrm{i}S[x]}, \end{aligned} \tag{2.9}$$

where $\mathcal{N}$ is a constant and $S[x]$ is the action. Here, the end points are held fixed and only the intermediate points are integrated over the entire space, which means that the transition amplitude is the sum over all paths connecting the two points weighted by the factor $e^{\mathrm{i}S[x]}$. The dominant contribution is the classical one, namely arising from paths which extremize the phase factor, i.e. the ones satisfying $\frac{\delta S[x]}{\delta x(t)}$.

Exercise 2 *Show that for a free particle, the transition amplitude (2.9) computes to*

$$\begin{aligned} U(t_f, x_f; t_i, x_i) &= \sqrt{\frac{m}{2\pi\mathrm{i}(t_f - t_i)}} e^{\mathrm{i}S[x_{cl}]}, \\ \lim_{t_f \to t_i} U(t_f, x_f; t_i, x_i) &= \delta(x_f - x_i), \end{aligned} \tag{2.10}$$

where x_{cl} denotes the classical trajectory, and that it solves the Schrödinger equation!

Exercise 3 *Repeat the calculation for the harmonic oscillator!*

2.2. Generating Functional

Lets us evaluate the matrix element of a product of operators of the form:

$$\begin{aligned} {}_H\langle x_f, t_f | X_H(t_1) X_H(t_2) | x_i, t_i \rangle_H &= \int dx_1 dx_2 x_1 x_2\, {}_H\langle x_f, t_f \,|\, x_1, t_1 \rangle_H\, {}_H\langle x_1, t_1 \,|\, x_2, t_2 \rangle_H \times \\ &\quad \times {}_H\langle x_2, t_2 \,|\, x_i, t_i \rangle_H \\ &= \mathcal{N} \int \mathcal{D}x x(t_1) x(t_2) e^{\mathrm{i}S[x]}, \end{aligned} \tag{2.11}$$

assuming $t_1 > t_2$ on the l.h.s. Since $x(t_1)$ and $x(t_2)$ commute, the path integral on the r.h.s. automatically incorporates time ordering. A similar derivation can be done for time ordered correlation functions between physical states (cf. [2], p.61).

By introducing the modified action

$$S[x,J] = S[x] + \int_{t_i}^{t_f} dt\, x(t)J(t)\,,$$

$$\langle x_f, t_f \,|\, x_i, t_i\rangle_J = \mathcal{N}\int \mathcal{D}x\, e^{\mathrm{i}S[x.J]} = \langle x_f, t_f | T e^{\mathrm{i}\int dt X(t)J(t)} | x_i, t_i\rangle\,, \tag{2.12}$$

we my rewrite the result above as

$$\mathcal{N}\int \mathcal{D}x x(t_1)x(t_2)e^{\mathrm{i}S[x]} = \frac{1}{\mathrm{i}^2}\frac{\delta^2\langle x_f, t_f \,|\, x_i, t_i\rangle_J}{\delta J(t_1)\delta J(t_2)}\bigg|_{J=0}\,, \tag{2.13}$$

which justifies the name "generating functional" for $\langle x_f, t_f \,|\, x_i, t_i\rangle_J$.

2.3. Path integral for Grassmann variables

Let θ_i with $i = 1, 2, \ldots, n$ be a set of Grassmann variables satisfying $\{\theta_i, \theta_j\} = 0$. This implies that if $f(\theta)$ is a function of just one Grassmann variable it has the Taylor expansion

$$f(\theta) = a + b\theta \tag{2.14}$$

since $\theta^2 = 0$.

A left derivative for Grassmann variables yields

$$\frac{\partial}{\partial\theta_i}(\theta_j\theta_k) = \delta_{ij}\theta_k - \delta_{ik}\theta_j\,. \tag{2.15}$$

Notice also that $\left\{\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j}\right\} = 0$, i.e. are nilpotent as well. Hence, $\left\{\frac{\partial}{\partial\theta_i}, \theta_j\right\} = \delta_{ij}$.

The notion of integration can also be generalized to Grassmann variables. Considering that the integral of a total derivative should vanish if we ignore surface terms and that a definite integral, being independent of the variable, must give zero upon differentiation. Since differentiation with respect to a Grassmann variable is nilpotent, it satisfies these properties and integration with respect to Grassmann variables can be naturally identified with differentiation, i.e.

$$\int d\theta f(\theta) = \frac{\partial f(\theta)}{\partial\theta}\,. \tag{2.16}$$

Therefore

$$\int d\theta = 0\,, \qquad \int d\theta\theta = 1\,, \tag{2.17}$$

and

$$\int d\theta f(\theta) = a\int d\theta' f(\theta'/a) \tag{2.18}$$

for $\theta' = a\theta$. Generalizing to several Grassmann variables we find

$$\int d\theta_1 \dots d\theta_n f(\theta_i) = (\det a_{ij}) \int d\theta'_1 \dots d\theta'_n f\left(a_{ij}^{-1}\theta'_j\right) \tag{2.19}$$

for $\theta'_i = a_{ij}\theta_j$ and $(\det a_{ij}) \neq 0$. Furthermore, a delta function can be defined as

$$\delta(\theta) = \theta = -\mathrm{i}\int d\zeta e^{\mathrm{i}\zeta\theta}\,, \tag{2.20}$$

where ζ is another Grassmann variable, since

$$\int d\theta\delta(\theta) = 1\,, \qquad \int d\theta\delta(\theta)f(\theta) = f(0)\,. \tag{2.21}$$

Finally, a Gaussian integral for Grassmann variables is given by

$$\begin{aligned}\int \prod_{i,j} d\theta_i^* \theta_j e^{-(\theta_i^* M_{ij}\theta_j + c_i^*\theta_i + \theta_i^* c_i)} &= \det M_{ij} \int \prod_{i,j} d\theta_i^* \theta_j e^{-\theta_i^{*\prime}\theta'_i + c_i^* M_{ij}^{-1} c_j} \\ &= \text{const. } \det M_{ij}\, e^{c_i^* M_{ij}^{-1} c_j}\,,\end{aligned} \tag{2.22}$$

assuming θ_i^* and θ_i as well as c_i^* and c_i are independent Grassmann variables.

Exercise 4 *Consider the following action for the fermionic harmonic oscillator:*

$$\begin{aligned}S[\psi,\bar\psi,\eta,\bar\eta] &= S[\psi,\bar\psi] + \int dt(\bar\eta\psi + \bar\psi\eta)\,, \\ S[\psi,\bar\psi] &= \int dt\left(\frac{\mathrm{i}}{2}(\bar\psi\dot\psi - \dot{\bar\psi}\psi) - \frac{\omega}{2}\left[\bar\psi,\psi\right]\right)\,,\end{aligned} \tag{2.23}$$

and compute the free fermion path integral!

2.4. Path integral for a relativistic scalar field theory

Let us consider the action of ϕ^4 theory

$$\begin{aligned}\mathcal{S}[\phi] &= \int d^4x \mathcal{L}(\phi, \partial_\mu\phi)\,, \\ \mathcal{L} &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4\,,\end{aligned} \tag{2.24}$$

and introduce appropriate source through the couplings

$$\mathcal{S}[\phi, J] = \mathcal{S}[\phi] + \int d^4x J\phi \tag{2.25}$$

leading to the e.o.m.

$$\frac{\delta\mathcal{S}[\phi,J]}{\delta\phi(x)} = \partial_\mu\partial^\mu\phi + m^2\phi + \frac{\lambda}{3!}\phi^3 - J = 0 \tag{2.26}$$

Furthermore, considering the limits $t_i \to -\infty$ and $t_f \to +\infty$, the vacuum to vacuum transition amplitude is given by

$$Z[J] = \langle 0\,|\,0\rangle_J = \mathcal{N}\int \mathcal{D}\phi e^{\mathrm{i}S[\phi,J]}\,. \tag{2.27}$$

For $\lambda = 0$ we easily derive

$$\begin{aligned} Z_0[J] &= \mathcal{N}\int \mathcal{D}\phi e^{\mathrm{i}S_0[\phi,J]} \\ &= \mathcal{N}\left(\det(\Box + m^2)\right)^{-\frac{1}{2}} e^{-\frac{\mathrm{i}}{2}\int d^4x d^4x' J(x)G(x-x')J(x')} \\ &= Z_0[0]e^{-\frac{\mathrm{i}}{2}\int d^4x d^4x' J(x)G(x-x')J(x')}\,, \end{aligned} \tag{2.28}$$

where the field ϕ was integrated out by shift and quadratic completion, $S_0[\phi] = S[\phi]\big|_{\lambda=0}$, and $G(x-x')$ is the Feynman propagator resp. the Green function we encountered earlier in Eqn. (1.20). It is therefore straightforward to see that the normalized time ordered correlation function

$$\langle 0|T\phi(x)\phi(x')|0\rangle\Big|_{\lambda=0} = \frac{1}{\mathrm{i}^2 Z_0[J]}\frac{\delta^2 Z_0[J]}{\delta J(x)\delta J(x')}\bigg|_{J=0} = \mathrm{i}G(x-x')\,. \tag{2.29}$$

The path integral $Z[J]$ of the full theory with $\lambda \neq 0$ cannot be evaluated in a closed form, but for weak coupling λ one can compute it perturbatively. By making the replacement $\phi(x) \to \frac{\delta}{\delta J(x)}$ in the interaction part of the action, we can derive the following power series in the coupling λ:

$$\begin{aligned} Z[J] &= \left(e^{-\frac{\mathrm{i}\lambda}{4!}\int d^4x\left(-\mathrm{i}\frac{\delta}{\delta J(x)}\right)^4}\right)\mathcal{N}\int \mathcal{D}\phi e^{\mathrm{i}S_0[\phi,J]} = \left(e^{-\frac{\mathrm{i}\lambda}{4!}\int d^4x\left(-\mathrm{i}\frac{\delta}{\delta J(x)}\right)^4}\right) Z_0[J] \\ &= \left[1 - \frac{\mathrm{i}\lambda}{4!}\int d^4x \frac{\delta^4}{\delta J^4(x)} + \frac{1}{2!}\left(-\frac{\mathrm{i}\lambda}{4!}\right)^2 \int d^4x \frac{\delta^4}{\delta J^4(x)}\int d^4x' \frac{\delta^4}{\delta J^4(x')} + \ldots\right] Z_0[J] \\ &= Z_0[0]\left[1 - \frac{\mathrm{i}\lambda}{4!}\int d^4x \frac{\delta^4}{\delta J^4(x)} + \ldots\right] e^{-\frac{\mathrm{i}}{2}\int d^4x_1 d^4x_2 J(x_1)G(x_1-x_2)J(x_2)}\,. \end{aligned} \tag{2.30}$$

Let us compute the 2-point correlation function to linear order in λ. For this we need

$$\begin{aligned} Z[J] = Z_0[0]e^{-\frac{\mathrm{i}}{2}\int d^4x_1 d^4x_2 J(x_1)G(x_1-x_2)J(x_2)}\Bigg[1 &+ \frac{\mathrm{i}\lambda}{8}G_F(0)G_F(0)\int d^4x \\ &+ \frac{\lambda}{4}G_F(0)\int d^4x d^4x_3 d^4x_4 G_F(x-x_3)J(x_3)G_F(x-x_4)J(x_4) \\ &- \frac{\mathrm{i}\lambda}{4!}\int d^4x d^4x_{3-6} G(x-x_3)J(x_3)G(x-x_4)J(x_4)G(x-x_5)J(x_5)G(x-x_6)J(x_6)\Bigg] \\ &+ \mathcal{O}(\lambda^2)\,, \end{aligned} \tag{2.31}$$

which follows from Eqn. (2.30) by explicit computation while consistently keeping only terms linear in λ. Notice that

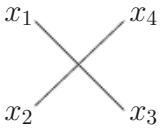
$$Z[0] = Z_0[0]\left(1 + \frac{\mathrm{i}\lambda}{8}G_F(0)G_F(0)\int d^4x\right) + \mathcal{O}(\lambda^2)\,, \tag{2.32}$$

is a divergent quantity which can be absorbed in the normalization $\mathcal{N}$. Continuing our computation of the two-point function to order λ, we arrive at

$$\begin{aligned}\langle 0|T\phi(x_1)\phi(x_2)|0\rangle &= \frac{1}{\mathrm{i}^2 Z(J)}\frac{\delta^2 Z(J)}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} \\ &= \mathrm{i}G_F(x_1-x_2) - \frac{\lambda}{2}G_F(0)\int d^4x G_F(x-x_1)G_F(x-x_2) + \mathcal{O}(\lambda^2)\,.\end{aligned} \tag{2.33}$$

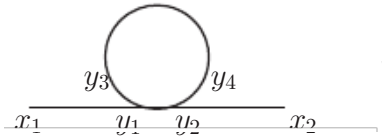
Similarly, we may compute the lowest order quantum correction to the 4-point function, which however involves numerous terms. Therefore, a systematic procedure keeping track of the perturbative expansion is helpful. Such a procedure is given by the so-called Feynman rules:

$$\mathrm{i}G_F(x_1-x_2)$$



$$\begin{aligned}V(x_1,x_2,x_3,x_4) &= \mathrm{i}\frac{\delta^4 S[\phi]}{\delta\phi(x_1)\delta\phi(x_2)\delta\phi(x_3)\delta\phi(x_4)} \\ &= -\mathrm{i}\lambda\int d^4x\,\delta^4(x-x_1)\delta^4(x-x_2)\delta^4(x-x_3)\delta^4(x-x_4)\,.\end{aligned} \tag{2.34}$$

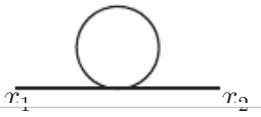
With these rules, supplemented by the rules that we must integrate over intermediate points in a graph and that if the internal part of a diagram has a symmetry one must divide by that symmetry factor[1], we find for the simplest 2-point diagram



$$\begin{aligned}&\frac{1}{2}\int d^4y_{1-4}\mathrm{i}G(x_1-y_1)\mathrm{i}G(y_2-x_2)\mathrm{i}G(y_3-y_4)V(y_1,y_2,y_3,y_4) \\ &= -\frac{\lambda}{2}G(0)\int d^4y G(x_1-y)G(y-x_2)\,.\end{aligned} \tag{2.35}$$

Hence, we see that the result of Eqn. (2.33) can be written diagrammatically as a simple sum:

$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle =$

$+$ 

$+\ldots$

[1] In order to compute the correct factor for a certain type of Feynman graph, one first counts the number of possibilities to connect the external lines to the vertices of a "pre-graph". That number is then multiplied by the number of possibilities to connect the vertices among each other, and the result is divided by 4! for every vertex and by $n!$ if n identical vertices are present. The latter $n!$-factor comes from the expansion of the exponential in the path integral. In the present 2-point graph, one arrives at $4\,.3/4! = 1/2$.

When computing the first order corrections to the 4-point function, we easily see that connected and disconnected graphs appear in the result of

$$\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle = \frac{1}{\mathrm{i}^4 Z[J]} \frac{\delta^4 Z[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\bigg|_{J=0} \tag{2.36}$$

$$= \quad x_1 \text{---} x_2 \;\; x_3 \text{---} x_4 \quad + \quad x_1 \text{---} x_3 \;\; x_2 \text{---} x_4 \quad + \quad x_1 \text{---} x_4 \;\; x_2 \text{---} x_3$$

$$+ 6 \text{ permutations of } \quad x_3 \text{---} x_4 \;\; x_1 \text{---} x_2 \quad + \quad x_1 \times x_4 \;\; x_2 \times x_3 \quad + \mathcal{O}(\lambda^2)\,,$$

i.e. the generating functional $Z[J]$ generates both types of graphs. In contrast, the $Z_c[J] := -\mathrm{i}\ln Z[J]$ generates only connected graphs and is hence called the generating functional of the connected graphs. For example, observe that

$$\begin{aligned}\frac{1}{\mathrm{i}}\frac{\delta^2 Z_c[J]}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} &= (-\mathrm{i})^2\left(\frac{1}{Z[J]}\frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)} - \frac{1}{Z^2[J]}\frac{\delta Z[J]}{\delta J(x_1)}\frac{\delta Z[J]}{\delta J(x_2)}\right)\bigg|_{J=0} \\ &= \langle 0|T\phi(x_1)\phi(x_2)|0\rangle - \langle 0|\phi(x_1)|0\rangle\langle 0|\phi(x_2)|0\rangle \\ &= \langle 0|T\phi(x_1)\phi(x_2)|0\rangle_c\,. \end{aligned} \tag{2.37}$$

In the present case, however, $\langle 0|\phi(x_1)|0\rangle = 0$. But for e.g. the 4-point functions there is a difference, and one can convince oneself that

$$\frac{1}{\mathrm{i}^4}\frac{\delta^4 Z_c[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\bigg|_{J=0} = \langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle_c = \quad x_1 \times x_4 \;\; x_2 \times x_3 \tag{2.38}$$

Exercise 5 *Explicitly compute the 4-point function up to order λ and show that Z_c generates only connected graphs while Z generates both connected and disconnected ones!*

At order λ^2 one discovers, that the connected diagrams generated by Z_c still contain two types of diagrams: the one particle reducible ones, which can be reduced to two connected diagrams by cutting an internal line, and the more fundamental one particle irreducible (1PI) graphs.

2.5. Effective action

Let us define the "classical field" $\phi_c(x)$ as[2]

$$\phi_c(x) = \langle 0|\phi(x)|0\rangle_J = \frac{1}{\mathrm{i}Z[J]}\frac{\delta Z[J]}{\delta J(x)} = \frac{\delta Z_c[J]}{\delta J(x)}\,. \tag{2.39}$$

[2]Note, that $\phi_c(x)$ is indeed a functional of the source J and in the present case of ϕ^4 theory vanishes for $J = 0$ — in general it becomes a constant.

The name "classical" is motivated by the following observation: Assuming that the measure of the path integral does not change under a redefinition of the field variable, we have

$$\delta Z[J] = \mathcal{N} \int \mathcal{D}\phi \, \mathrm{i} \int d^4x \delta\phi(x) \frac{\delta S[\phi, J]}{\delta\phi(x)} e^{\mathrm{i}S[\phi,J]} = 0 \,, \tag{2.40}$$

since Z is independent of ϕ, and it follows that the Euler-Lagrange equations hold as an expectation value equation (Ehrenfest's theorem):

$$\langle 0 | \frac{\delta S[\phi, J]}{\delta\phi(x)} | 0 \rangle_J = 0 \,. \tag{2.41}$$

Considering the generic form $-\frac{\delta S[\phi,J]}{\delta\phi(x)} = F(\phi(x)) - J(x)$ we find

$$\begin{aligned} -\mathcal{N} \int \mathcal{D}\phi \frac{\delta S[\phi, J]}{\delta\phi(x)} e^{\mathrm{i}S[\phi,J]} = 0 \,, \qquad & \text{or} \\ \left[F\left(-\mathrm{i}\frac{\delta}{\delta J(x)} \right) - J(x) \right] Z[J] = 0 \,, \qquad & \text{or} \\ F\left(\frac{\delta Z_c[J]}{\delta J(x)} - \mathrm{i}\frac{\delta}{\delta J(x)} \right) - J(x) = 0 \,. & \end{aligned} \tag{2.42}$$

Restoring $\hbar$ (which have set to 1) in this equation and using Eqn. (2.39), we get

$$F\left(\phi_c(x) - \mathrm{i}\hbar \frac{\delta}{\delta J(x)} \right) - J(x) = 0 \,, \tag{2.43}$$

which in the classical limit $\hbar \to 0$ reduces to the form of the classical Euler-Lagrange equations.

The relation (2.39) indicates that the variables $J(x)$ and $\phi_c(x)$ are in some sense conjugate variables. This motivates the definition of a new functional through the Legendre transformation

$$\Gamma[\phi_c] := Z_c[J] - \int d^4x J(x) \phi_c(x) \,. \tag{2.44}$$

Clearly, $\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} = -J(x)$ which reminds us of $\frac{\delta S[\phi]}{\delta\phi(x)} = -J(x)$, and therefore $\Gamma[\phi_c]$ is known as the "effective action functional". Note that

$$\int d^4z \frac{\delta^2 Z_c[J]}{\delta J(y) \delta J(z)} \frac{\delta^2 \Gamma[\phi_c]}{\delta\phi_c(z) \delta\phi_c(x)} = -\delta(x - y) \,, \tag{2.45}$$

which follows from $\frac{\delta}{\delta J(y)} \frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} = -\delta(x-y)$ using the chain rule and the definition of ϕ_c. Introducing the abbreviations

$$\Gamma^{(n)} := \frac{\delta^n \Gamma[\phi_c]}{\delta\phi_c(x_1) \cdots \delta\phi_c(x_n)} \,, \qquad Z_c^{(n)} := \frac{\delta^n Z_c[J]}{\delta J(x_1) \cdots \delta J(x_n)} \,, \tag{2.46}$$

one can show that $\Gamma^{(n)}\big|_{\phi_c}$, i.e. $\Gamma^{(n)}$ evaluated at $\phi_c[J=0]$, which for ϕ^4-theory is 0, is the 1PI n-point vertex function. Hence, $\Gamma[\phi_c]$ is also known as the 1PI generating functional.

For example, since $Z_c^{(2)}\big|_{J=0} = -G^{(2)}$, Eqn. (2.45) tells us that $\Gamma^{(2)}\big|_{\phi_c}$ is the inverse of the full propagator at every order of perturbation theory. Hence

$$\begin{aligned}\Gamma^{(2)}\big|_{\phi_c} &= \Gamma_0^{(2)} - \Sigma\,,\\ G^{(2)} &= \frac{1}{G_F^{-1} - \Sigma} = G_F + G_F\Sigma G_F + G_F\Sigma G_F\Sigma G_F + \dots\,,\end{aligned} \tag{2.47}$$

using Eqn. (2.45) in a short-hand notation in the second step. Furthermore, it follows from varying Eqn. (2.45) with respect to $J(\omega)$ that

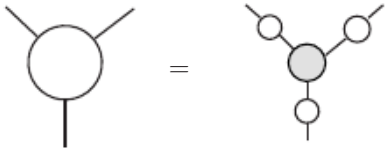
$$\int d^4z \frac{\delta^3 Z_c[J]}{\delta J(\omega)\delta J(y)\delta J(z)}\frac{\delta^2\Gamma[\phi_c]}{\delta\phi_c(z)\delta\phi_c(x)} = -\int d^4z d^4\sigma \frac{\delta^2 Z_c[J]}{\delta J(y)\delta J(z)}\frac{\delta^3\Gamma[\phi_c]}{\delta\phi_c(z)\delta\phi_c(x)\delta\phi_c(\sigma)}\frac{\delta^2 Z_c[J]}{\delta J(\sigma)\delta J(\omega)} \tag{2.48}$$

and hence using (2.45) once more one finds

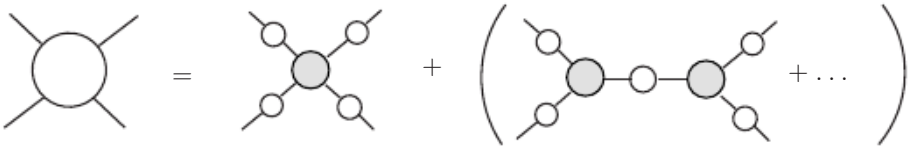
$$\frac{\delta^3 Z_c[J]}{\delta J(\omega)\delta J(y)\delta J(z)} = \int d^4x' d^4y' d^4z' \frac{\delta^2 Z_c[J]}{\delta J(x)\delta J(x')}\frac{\delta^2 Z_c[J]}{\delta J(y)\delta J(y')}\frac{\delta^2 Z_c[J]}{\delta J(z)\delta J(z')}\frac{\delta^3\Gamma[\phi_c]}{\delta\phi_c(x')\delta\phi_c(y')\delta\phi_c(z')}\,, \tag{2.49}$$

or in short hand notation

$$Z_c^{(3)} = Z_c^{(2)} Z_c^{(2)} Z_c^{(2)} \Gamma^{(3)}\,. \tag{2.50}$$



Similar relations can be derived for the 4 or n-point Green functions by varying Eqn. (2.45), i.e. for the former one has in graphical notation:



2.6. S matrix

Let us define an incoming free field $\phi_{\text{in}}(x) := \lim_{x_0\to-\infty}$ assuming an adiabatic vanishing of the source j at $x_0 \to \pm\infty$. Similarly, an outgoing free field is then $\phi_{\text{out}}(x) := \lim_{x_0\to+\infty}$. The idea is, that a free incoming field propagates in time, a source is adiabatically switched on, the field interacts, and the source is adiabatically turned off leaving a once more free outgoing field. This construction, of course takes place in a given Hilbert space, and one may try to find the unitary operator S which connects the in- and out-fields (and states):

$$\phi_{\text{out}}(x) = S^{-1}\phi_{\text{in}}S\,, \qquad\qquad |\text{out}\rangle = S^{-1}|\text{in}\rangle\,. \tag{2.51}$$

In general, one is of course interested in the situation where an initial configuration of particles α ends up as a final configuration β after a scattering process. The scattering amplitude for this is denoted

$$S_{\alpha\beta} = {}_{\rm out}\langle\beta\,|\,\alpha\rangle_{\rm in} = {}_{\rm in}\langle\beta|S|\alpha\rangle_{\rm in}\,, \tag{2.52}$$

and its square measures the probability for the process in question. If for example $|\alpha\rangle_{\rm in}$ consists of two scalar particles with momenta p_1 and p_2, then $|\alpha\rangle_{\rm in} = a^\dagger_{\rm in}(p_1)a^\dagger_{\rm in}(p_2)|0\rangle$ where $a^\dagger$ are creation operators. Obviously, invariance of the vacuum demands that $S_{00} = 1$.

In Eqn. (1.19) we have already given a general solution to the classical e.o.m. for a scalar field. Using the expressions for advanced and retarded Green functions of (1.21) we may write

$$\phi(x) = \phi_{\rm in}(x) + \int d^4x G_{\rm ret}(x,x')j(x') = \phi_{\rm out}(x) + \int d^4x G_{\rm adv}(x,x')j(x')\,, \tag{2.53}$$

and consider the weak asymptotic condition $\lim\limits_{t\to\pm\infty}\langle a|\phi(x)|b\rangle = \langle a|\phi_{\substack{\rm out\\ \rm in}}(x)|b\rangle$. (Note, that we are considering quantized fields ϕ in this section.)

Now, the inhomogeneity j above is due to the interaction part, i.e. should be replaced by

$$j(x') \to \frac{\partial\mathcal{L}_{\rm int}}{\partial\phi(x')} = K_{x'}\phi(x')\,, \tag{2.54}$$

where the last step follows from the e.o.m and $K = (\Box + m^2)$ is the Klein-Gordon operator.

Next, we define the functional

$$I[J] = Te^{{\rm i}\int d^4x J(x)\phi(x)}\,, \tag{2.55}$$

whose vacuum expectation value is the generating functional $Z[J]$, i.e. $\langle 0|I[J]|0\rangle = Z[J]$. It then follows from (2.53), (2.51) and $S^{-1} = S^\dagger$ that

$$\begin{aligned}
\phi_{\rm out}I - I\phi_{\rm in} &= {\rm i}\int d^4x\left(G_{\rm ret}(x,x') - G_{\rm adv}(x,x')\right)K_{x'}\frac{\delta I[J]}{\delta J(x')}\,, \qquad \text{and}\\
[\phi_{\rm in}(x), SI[J]] &= {\rm i}\int d^4x\left(G_{\rm ret}(x,x') - G_{\rm adv}(x,x')\right)K_{x'}\frac{\delta SI[J]}{\delta J(x')}\,.
\end{aligned} \tag{2.56}$$

Now one can show that upon promoting $\phi^\pm$ in Eqn. (1.22) to creation/annihilation operators, that the expression $G_{\rm adv} - G_{\rm ret}$ coincides with the commutator of free scalar fields $[\phi_{\rm in}(x), \phi_{\rm in}(x')]$. Employing furthermore the Baker-Campbell-Hausdorff formula in the form $\left[A, e^B\right] = [A,B]\,e^B$ for the case where $[A,B]$ is a c-number, the solution

$$SI[J] = \exp\left(\int d^4x\phi_{\rm in}(x)K\frac{\delta}{\delta J(x)}\right)F[J]\,,$$

where $F[J]$ is some functional of J, suggests itself. Taking the vacuum expectation value thereof and considering $\langle 0|S = \langle 0|$ in the absence of external fields, we find (for normal ordered SI):

$$\langle 0|SI[J]|0\rangle = F[J]\,, \qquad \text{and} \quad \langle 0|SI[J]|0\rangle = \langle 0|I[J]|0\rangle = Z[J]\,. \tag{2.57}$$

Hence, we arrive at the so-called reduction formula

$$S =: \exp\left(\int d^4x \, \phi_{\text{in}}(x) K_x \frac{\delta}{\delta J(x)} \right) : Z[J] \Big|_{J=0} , \tag{2.58}$$

where ":" denotes normal ordering. The above formula means, that for a given Feynman graph, one has to amputate all external legs and replace them by the incoming free field wave functions ϕ_{in}. (The operator K converts external propagators into δ-functions.) Hence, the n-particle S-matrix element is

$$S_n(x_1, \ldots, x_n) = \prod_i \phi(x_i) K_{x_i} G(x_1, \ldots, x_n) , \tag{2.59}$$

where $G(x_1, \ldots, x_n)$ denotes the n-particle Green function.

2.7. Euclidean path integrals

By analytic continuation of all integrals to imaginary time in the complex t-plane, i.e. $t \to t' := -\mathrm{i}\tau$ where $\tau \in \mathbb{R}$, one arrives at Euclidean path integrals. This "Wick-rotation" can be useful to simplify certain computations, taking into account the famous Osterwalder–Schrader theorem which basically states that if a quantum field theory exists in Euclidean space, it also exists in the Wick-rotated Minkowski space-time. Of course, one must take care not to cross any poles when doing a Wick-rotation. For example, consider

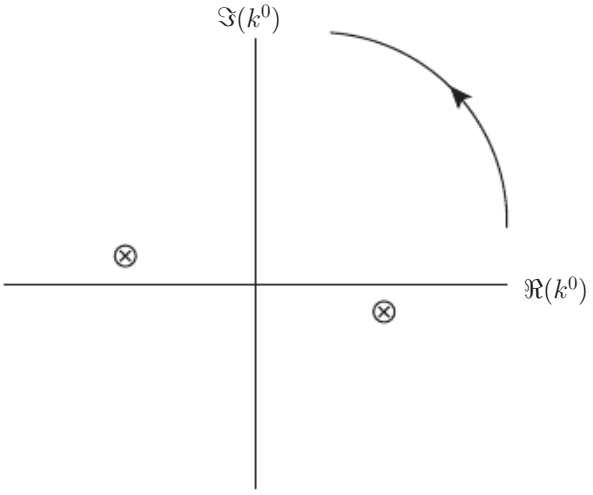


Figure 2.1.: Wick-rotation

the following propagator (of the harmonic oscillator):

$$G_F(k) = \lim_{\epsilon \to 0^+} \frac{1}{\sqrt{2\pi}} \frac{1}{k + \omega - \mathrm{i}\epsilon} \frac{1}{k - \omega + \mathrm{i}\epsilon} , \tag{2.60}$$

which has poles at $k = -\omega + \mathrm{i}\epsilon$ and $k = \omega - \mathrm{i}\epsilon$. Hence, for the analytic continuation from $\Re k^0$ to $\Im k^0$ one must rotate anticlockwise, i.e. $k^0 \to k'^0 = \mathrm{i}\kappa$, as indicated in Figure 2.1. Since

we can represent $k^0 \to \mathrm{i}\frac{\partial}{\partial t}$, it follows that in the complex t-plane the consistent rotation will be $t \to t' = -\mathrm{i}\tau$ as stated above.

Euclidean path integrals can in fact be interpreted in a statistical context. Let us make this clearer by reviewing some facts of statistical ensembles: For a given ensemble (e.g. a system of harmonic oscillators), the value of any observable quantity averaged over the entire ensemble will take the form

$$\langle A \rangle = \sum_n p_n \langle n|A|n\rangle = \sum_n p_n A_n \,, \tag{2.61}$$

where p_n is the probability of finding a system in the ensemble to be in an energy eigenstate $|n\rangle$. In fact, there are two kinds of averaging involved here: the average in a quantum state (the expectation value) and the averaging with respect to the probability distribution of systems in the ensemble. As p_n is a probability, it has to satisfy the conditions

$$0 \leq p_n \leq 1 \,, \qquad\qquad \sum_n p_n = 1 \,. \tag{2.62}$$

If we consider a thermodynamic ensemble interacting with a large heat bath, and assume we have waited long enough to achieve thermal equilibrium, then the probability distribution is given by the Maxwell-Boltzmann distribution

$$p_n = \frac{1}{Z} e^{-\beta E_n} \,, \qquad\qquad \beta := \frac{1}{k_B T} \,, \tag{2.63}$$

where E_n is the energy of the nth quantum state, k_B is Boltzmann's constant, and T the temperature of the system. Using the conditions for p_n, one easily determines the normalization Z:

$$Z(\beta) = \sum_n e^{-\beta E_n} = \sum_n \langle n|e^{-\beta H}|n\rangle = \mathrm{Tr} e^{-\beta H} \,. \tag{2.64}$$

In fact, $Z(\beta)$ is known as the partition function of the system and plays the most fundamental role in deriving the thermodynamic properties of the system.

For such a thermodynamical ensemble, the thermodynamic average according to (2.61) is given by

$$\langle A \rangle_\beta = \frac{1}{Z(\beta)} \sum_n \langle n|e^{-\beta H} A|n\rangle = \frac{\mathrm{Tr}\left(e^{-\beta H} A\right)}{\mathrm{Tr} e^{-\beta H}} \,. \tag{2.65}$$

In particular, the average energy computes to

$$\langle H \rangle_\beta = U = -\frac{1}{Z(\beta)} \frac{\partial Z(\beta)}{\partial \beta} = -\frac{\partial \ln Z(\beta)}{\partial \beta} \,, \tag{2.66}$$

and entropy is given by

$$\begin{aligned} \langle \ln p \rangle_\beta = S &= -\sum_n p_n \ln p_n = \sum_n p_n \left(\beta E_n + \ln Z(\beta)\right) = \beta U + \ln Z(\beta) \\ &= -\beta^2 \frac{\partial}{\partial \beta} \left(\frac{1}{\beta} \ln Z(\beta)\right) \,. \end{aligned} \tag{2.67}$$

The free energy can hence be written as

$$\begin{aligned} F(\beta) &= U - k_B T S \\ &= -\frac{1}{\beta} \ln Z(\beta) \,, \end{aligned} \tag{2.68}$$

and hence the partition function takes a particularly simple form when expressed in terms of free energy:

$$Z(\beta) = e^{-\beta F(\beta)} \,. \tag{2.69}$$

To sum up, a thermodynamical statistical ensemble can be described by a Euclidean path integral where the Euclidean time interval plays the role of temperature and the partition function is identified with the transition amplitude. In general, one uses Euclidean path integrals to study phase transitions of quantum systems — an example from high energy physics would be a quark-gluon plasma.

2.8. Path integrals for gauge theories

Our starting point is the generating functional

$$Z[J_\mu] = e^{\mathrm{i}Z_c[J_\mu]} = \mathcal{N} \int \mathcal{D}A_\mu e^{\mathrm{i}S_J[A_\mu]} \,, \tag{2.70}$$

where

$$\begin{aligned} S_J[A_\mu] &= \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + j_\mu A^\mu \right) \\ &= \frac{1}{2} \int d^4x \int d^4x' A_\mu(x) \mathcal{O}^{\mu\nu}(x - x') A_\nu(x') + \int d^4x J_\mu(x) A^\mu(x) \,, \\ \mathcal{O}^{\mu\nu}(x - x') &= (\Box \eta^{\mu\nu} - \partial^\mu \partial^\nu) \, \delta^4(x - x') \,, \end{aligned} \tag{2.71}$$

neglecting surface terms. The path integral over the gauge fields A_μ would lead to $\det \mathcal{O}$. Unfortunately, this integral does not exist since $\mathcal{O}$ is a projection operator (cf. Section 1.2.2). In fact, the source of this difficulty is the invariance of the above action under a gauge transformation

$$\begin{aligned} A_\mu \to A_\mu^{(\alpha)} &= U A_\mu U^{-1} + \mathrm{i} U^{-1} (\partial_\mu U) \,, \\ U(\alpha) &= e^{-\mathrm{i}\alpha(x)} \,. \end{aligned} \tag{2.72}$$

All $A_\mu^{(\alpha)}$ that can be obtained from a certain A_μ by making a gauge transformation with arbitrary $\alpha(x)$ are said to lie on an "orbit" in group space (in this case the Abelian $U(1)$ group). Since the action is invariant on such orbits, the generating functional $Z[J_\mu]$ is proportional to the "volume" of the orbits denoted by $\int \prod_x d\alpha(x)$ (resp. the group invariant Haar measure $\prod_x dU(x)$ in the non-Abelian case). This infinite factor needs to be extracted before doing any calculations, and this is best done by the method of Faddeev and Popov: The idea is to "fix" the gauge freedom and to integrate over each orbit only once, i.e. one chooses a hypersurface defined by a "gauge condition" $F(A_\mu) = 0$, which intersects each

gauge orbit only once. Even if A_μ does not satisfy this condition, one can find a gauge transformed $A_\mu^{(\alpha)}$ which does, i.e. $F(A_\mu^{(\alpha)}) = 0$ then has a unique solution[3] for $\alpha(x)$.

Now the trick due to Faddeev and Popov to extract out the infinite gauge volume factor relies on the insertion of

$$\Delta_{FP}[A_\mu] \int \prod_x d\alpha(x) \delta\left(F(A_\mu^{(\alpha)})\right) = 1\,, \tag{2.73}$$

into the path integral. Note that $\Delta_{FP}[A_\mu]$ is a gauge invariant quantity because its inverse is. (The measure in the group space is invariant under a gauge transformation.) We make an inverse gauge transformation $A_\mu \to A_\mu^{(-\alpha)}$, absorb the gauge volume into the normalization $\mathcal{N}$ and arrive at

$$Z[J_\mu] = \mathcal{N} \int \mathcal{D}A_\mu \Delta_{FP}[A_\mu] \delta\left(F(A_\mu)\right) e^{\mathrm{i}S_J[A_\mu]}\,. \tag{2.74}$$

In order to determine $\Delta_{FP}[A_\mu]$, notice first that

$$\begin{aligned} \Delta_{FP}^{-1}[A_\mu] &= \int \prod_x d\alpha(x) \delta\left(F(A_\mu^{(\alpha)})\right) = \int \prod_x dF \delta\left(F(A_\mu^{(\alpha)})\right) \left(\det \frac{\delta\alpha}{\delta F}\right) \\ &= \det\left(\frac{\delta\alpha}{\delta F}\right)\Big|_{F(A_\mu^{(\alpha)})=0}\,. \end{aligned} \tag{2.75}$$

Thus, we find the Faddeev-Popov determinant

$$\Delta_{FP}[A_\mu] = \det\left(\frac{\delta F(A_\mu^{(\alpha)})}{\delta\alpha}\right)\Big|_{\alpha(x)=0}\,, \tag{2.76}$$

which can be thought of as the Jacobian that goes with a particular gauge choice.

In a further step, we generalize the above to $F(A_\mu(x)) = f(x)$, where $f(x)$ is independent of A_μ. Since physical quantities are independent of f, we may multiply the generating functional by a weight factor and integrate over all $f(x)$ to arrive at

$$\begin{aligned} Z[J_\mu] &= \mathcal{N} \int \mathcal{D}A_\mu \mathcal{D}f \Delta_{FP}[A_\mu] \delta\left(F(A_\mu) - f(x)\right) e^{\mathrm{i}S_J[A_\mu]} e^{-\frac{\mathrm{i}}{2\xi}\int d^4x (f(x))^2} \\ &= \mathcal{N} \int \mathcal{D}A_\mu \Delta_{FP}[A_\mu] e^{\mathrm{i}S_J[A_\mu] - \frac{\mathrm{i}}{2\xi}\int d^4x (F(A_\mu(x)))^2}\,, \end{aligned} \tag{2.77}$$

where ξ is known as the gauge fixing parameter. In fact, this new term in the action can equivalently be seen as the result of integrating out a new multiplier field b. Namely, observe that

$$\mathcal{N} \int \mathcal{D}b\, e^{\mathrm{i}\int d^4x \left(F(A_\mu) b + \frac{\xi}{2} b^2\right)} = \mathcal{N}' e^{-\frac{\mathrm{i}}{2\xi}\int d^4x (F(A_\mu))^2}\,. \tag{2.78}$$

Finally, we can also write the Faddeev-Popov determinant in a path integral form by introducing unphysical "ghost" fields c and $\bar{c}$:

$$\Delta_{FP}[A_\mu] = \int \mathcal{D}\bar{c}\mathcal{D}c e^{-\mathrm{i}\int d^4x \int d^4x' \bar{c}(x) \left(\frac{\delta F(A_\mu^{(\alpha)}(x))}{\delta\alpha(x')}\right)_{\alpha=0} c(x')}\,. \tag{2.79}$$

[3] One should mention at this point, that such a hypersurface cannot be found in the non-Abelian case in the sense that any hypersurface will intersect gauge orbits more than once. This is commonly referred to as the Gribov ambiguity.

Thus the complete action (without sources) for this choice of gauge condition reads

$$S[A_\mu, b, \bar{c}, c] = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (\partial_\mu A^\mu) b + \frac{\xi}{2} b^2 + \partial_\mu \bar{c}(x) \partial^\mu c(x) \right) , \tag{2.80}$$

dropping surface terms once more. In the present model, the ghosts do not interact with the gauge fields and can hence be neglected in explicit loop computations, however, in non-Abelian gauge theories this is no longer true.

Exercise 6 *Show that the gauge field propagator in momentum space derived from the action* (2.80) *is given by*

$$G_{\mu\nu}(p) = \frac{-\mathrm{i}}{p^2} \left(\eta_{\mu\nu} - (1-\xi) \frac{p_\mu p_\nu}{p^2} \right) . \tag{2.81}$$

Why is it impossible for the auxiliary field b to appear in Feynman graphs?

BRST invariance.

Since we have actually "merely" inserted a factor of unity, Eqn. (2.73), into the path integral, the physical content of the theory has not changed. On the other hand, the action Eqn. (2.80) no longer exhibits the gauge symmetry due to the gauge fixing terms. However, this action exhibits a new global nilpotent fermionic symmetry, which in some sense "remembers" the gauge symmetry of the original theory. It is called the BRST symmetry[4], and in our case the according symmetry transformations which leave the action above invariant are given by

$$\begin{aligned} sA_\mu &= \partial_\mu c\,, & sc &= 0\,, \\ s\bar{c} &= b\,, & sb &= 0\,, \\ s^2 &= 0\,. \end{aligned} \tag{2.82}$$

If one repeats the same Faddeev-Popov procedure for non-Abelian $SU(N)$ gauge fields where

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g\,[A_\mu, A_\nu] = F^a_{\mu\nu} T^a\,, \\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu\,, \\ S_{\mathrm{inv}} &= \int d^4x \frac{-1}{2} \mathrm{Tr} F_{\mu\nu} F^{\mu\nu} = \int d^4x \frac{-1}{4} F^a_{\mu\nu} F^{a,\mu\nu}\,, \end{aligned} \tag{2.83}$$

f^{abc} are the antisymmetric structure constants, and T^a are the generators of the $SU(N)$ group, one arrives at

$$\begin{aligned} S(A_\mu, b, \bar{c}, c) &= \int d^4x \left(-\frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} + (\partial^\mu A^a_\mu) b^a + \frac{\xi}{2} (b^a)^2 + \partial^\mu \bar{c}^a (D_\mu c)^a \right) , \\ D^{ab}_\mu &= \delta^{ab} \partial_\mu + g f^{abc} A^c_\mu\,. \end{aligned} \tag{2.84}$$

[4] This symmetry was discovered in 1975 by C. Becchi, A. Rouet and R. Stora, and independently by I. V. Tyutin in the same year.

This action is invariant under the BRST transformations

$$\begin{aligned} sA^a_\mu &= (D_\mu c)^a\,, & sc^a &= -\frac{g}{2} f^{abc} c^b c^c\,, \\ s\bar{c}^a &= b^a\,, & sb^a &= 0\,, \\ s^2 &= 0\,. \end{aligned} \tag{2.85}$$

Note, that as before the BRST transformation of the gauge field has the same form as the according infinitesimal gauge transformation, but with the ghost $c(x)$ replaced by some (bosonic) function $\alpha^a(x)$, i.e.

$$\begin{aligned} A_\mu(x) &\to U^{-1}(x) A_\mu(x) U(x) + \frac{\mathrm{i}}{g} U^{-1}(x) \partial_\mu U(x) \approx A_\mu(x) + D_\mu \alpha(x)\,, \\ U(x) &= e^{-\mathrm{i} g \alpha^a(x) T^a}\,. \end{aligned} \tag{2.86}$$

Furthermore, in the non-Abelian case the field tensor transforms covariantly under an infinitesimal gauge transformation, i.e. as

$$F_{\mu\nu} \to U^{-1} F_{\mu\nu} U \approx F_{\mu\nu} - \mathrm{i} g \left[F_{\mu\nu}, \alpha(x)\right]\,, \tag{2.87}$$

(in contrast to the Abelian case where it is invariant). However, the trace of two field tensors $\mathrm{Tr} F_{\mu\nu} F^{\mu\nu}$ (and hence the invariant part of the action) is gauge invariant due to cyclic invariance of the trace.

As a final comment to this section, we mention that the physical space of the gauge theory can be identified as satisfying

$$Q_{\mathrm{BRST}} |\mathrm{phys}\rangle = 0\,, \tag{2.88}$$

where Q_{BRST} is the charge constructed from the Noether current of the BRST symmetry. Additionally, physical states are free of ghosts, as should already be clear by how the ghosts were introduced into the theory. Since $Q^2_{\mathrm{BRST}} = 0$, our Hilbert space actually decomposes into three subspaces: one where $Q_{\mathrm{BRST}}|\psi\rangle \neq 0$, one where states may be written as $|\psi\rangle = Q_{\mathrm{BRST}}|\psi'\rangle$ and hence are annihilated by Q_{BRST}, and the physical one above but where states cannot be written as in the second subspace. Furthermore, the S-matrix of the physical subspace must be unitary so as not to mix physical and unphysical asymptotic states. This is of course the case, as can be shown.

In the Abelian case, condition (2.88) implies the Gupta-Bleuler condition $\partial^\mu A_\mu |\mathrm{phys}\rangle = 0$ for the gauge fixing we considered above. One may also verify through explicit computation that $\frac{\partial Z[J]}{\partial \xi} = 0$ using Eqn. (2.88) and the fact that the vacuum belongs to the physical Hilbert space.

Ward identities.

BRST invariance of the theory leads to various relations between Feynman graphs (and hence scattering amplitudes) called Ward identities. These play an important role when it comes to proving renormalizability, and additionally provide various consistency checks when doing explicit computations.

In the derivation above, we have introduced several new fields into the gauge theory for which we need to introduce sources as well. we consider the non-Abelian case (of which the Abelian theory is a simpler special case), and write

$$S_J[A, b, \bar{c}, c] = S[A, b, \bar{c}, c] + \int d^4x \left(j^{a,\mu} A^a_\mu + j^a b^a + \bar{\eta}^a c^a - \bar{c}^a \eta^a \right)\,. \tag{2.89}$$

A BRST transformation of this action inside the path integral leads to

$$\begin{aligned} sS_J[A,b,\bar{c},c] &= \int d^4x \left(j^{a,\mu}(D_\mu c)^a + \frac{g}{2} f^{abc} \bar{\eta}^a c^b c^c - b^a \eta^a \right) , \\ sZ[J] = 0 &= \mathcal{N} \int \mathcal{D}A\mathcal{D}b\mathcal{D}\bar{c}\mathcal{D}c \, \mathrm{i} \left(sS_J[A,b,\bar{c},c] \right) e^{\mathrm{i}S_J[A,b,\bar{c},c]} \\ &= \int d^4x \left(j^{a,\mu} \langle (D_\mu c)^a \rangle + \bar{\eta}^a \langle \frac{g}{2} f^{abc} c^b c^c \rangle - \eta^a \langle b^a \rangle \right) . \end{aligned} \tag{2.90}$$

If we introduce two further external sources for the non-linear BRST transformations

$$\begin{aligned} S_{\text{ext}} &= \int d^4x \left(K^{a,\mu} (D_\mu c)^a - K^a \frac{g}{2} f^{abc} c^b c^c \right) , \\ S_{J,K}[A,b,\bar{c},c] &:= S_J[A,b,\bar{c},c] + S_{\text{ext}} , \end{aligned} \tag{2.91}$$

we can rewrite (2.90) as

$$\int d^4x \left(j^{a,\mu}(x) \frac{\delta Z_c[J,K]}{\delta K^{a,\mu}(x)} - \bar{\eta}^a(x) \frac{\delta Z_c[J,K]}{\delta K^a(x)} - \eta^a(x) \frac{\delta Z_c[J,K]}{\delta j^a(x)} \right) = 0 . \tag{2.92}$$

From this "master equation" one can derive all identities relating the connected Green functions of the theory. A Legendre transformation allows to rewrite the master equation in terms of the effective action, i.e. with

$$\Gamma[A,b,\bar{c},c,K] = Z_c[J,K] - \int d^4x \left(j^{a,\mu} A^a_\mu + j^a b^a + \bar{\eta}^a c^a - \bar{c}^a \eta^a \right) , \tag{2.93}$$

(where we dropped the subscripts c of the classical fields — cf. Section 2.5) it follows that

$$- \int d^4x \left(\frac{\delta \Gamma}{\delta A^a_\mu(x)} \frac{\delta \Gamma}{\delta K^{a,\mu}(x)} + \frac{\delta \Gamma}{\delta c^a(x)} \frac{\delta \Gamma}{\delta K^a(x)} + \frac{\delta \Gamma}{\delta \bar{c}^a(x)} b^a \right) = 0 . \tag{2.94}$$

In this form, the master equation is commonly referred to as the Slavnov-Taylor identity, and it allows to derive all relation between the 1PI vertices resulting from BRST invariance of the theory. Note, that at tree level (i.e. in the classical approximation) where $\Gamma[\,\phi_c] = S[\phi]$, the Slavnov-Taylor identity is just another way of writing $sS[\phi] = 0$.

3. Renormalization

3.1. Regularization and power counting

In general, perturbative computations in quantum field theories involve distributions multiplied with one another. These expressions are mathematically ill-defined and hence usually lead to divergences. In order to handle such expressions, the common strategy is to regularize the expressions in order to be able to split off the divergent parts which then may be absorbed by redefinition of the parameters of the Lagrangian. The latter process is referred to as renormalization.

Let us illustrate this procedure by considering a scalar quantum field theory, namely ϕ^4 theory: From Eqn. (1.20) we see that the propagator diverges for $x = x'$, i.e.

$$G(0) = -\int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2} \,. \tag{3.1}$$

Since there are four powers of p in the numerator and two in the denominator, this integral is expected to diverge quadratically at large p (hence the term "ultraviolet/UV divergence"). Similarly, the square of a propagator is a divergent quantity as well:

$$[G(x - x')]^2 = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-x')} \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2)\left((p-k)^2 - m^2\right)} \,, \tag{3.2}$$

which diverges logarithmically for large k.

Power counting.

The superficial degree of divergence of an arbitrary Feynman graph can be inferred by considering the properties of the Feynman rules for propagators and vertices of the model in question. For ϕ^4 theory, for example, one has one propagator behaving like $1/k^2$ for large momenta, hence reducing the degree of divergence by 2. Furthermore, every 4-dimensional loop integral raises the degree of divergence by 4. Hence,

$$d(\gamma) = 4L - 2I \,, \tag{3.3}$$

where L denotes the number of loops and I denotes the number of internal lines (i.e. propagators). Since there are I internal momenta as well as momentum conservation at each vertex and finally also overall momentum conservation, the number of independent momenta, which is L, is given by the relation

$$L = I - (V - 1) \,, \tag{3.4}$$

where V denotes the number of vertices in the graph. Finally, one needs relations between the number of vertices and the number of legs. External legs denoted by E count once,

whereas internal legs I count twice as they are always connected to two vertices (or two legs at one vertex). Thus, one finds the following relation:

$$4V = E + 2I\,. \tag{3.5}$$

Putting all pieces together, we hence find

$$\begin{aligned} d(\gamma) &= 4 + 2I - 4V \\ &= 4 - E\,. \end{aligned} \tag{3.6}$$

Notice, that the simultaneous elimination of I and V was only possible in 4-dimensional space-time. Furthermore, the mass dimension of the scalar field ϕ is 1, as can be seen from the action Eqn. (2.24). In general, the field dimension will be related to the numerical prefactor of E in the according power counting formula.

Exercise 7 *Check this claim by deriving the according power counting for QED!*

The power counting formula for scalar ϕ^4 theory Eqn. (3.6) tells us, that any correction to the two-point function can at most diverge quadratically ($E = 2 \;\rightarrow\; d(\gamma) = 2$), and any correction to the four point function can at most diverge logarithmically ($E = 4 \;\rightarrow\; d(\gamma) = 0$) independent of the number of loops. The fact that only a finite number of n-point functions (namely two) exhibit divergences is an important requirement for renormalizability of the model, since we only have a limited number of parameters we may redefine to absorb these divergences. In order to do so, non-existent divergent integrals must first be regularized.

Various regularizations schemes are known, and we start by reviewing Pauli-Villars regularization applied to the example above and then compare to dimensional regularization. Both schemes preserve translation and rotation invariance of the theory.

Pauli-Villars regularization.

This regularization scheme was introduced by in 1949 by W. Pauli and F. Villars and is based on substituting the Feynman propagator $G_F(x - x', m)$ by an expression of the form

$$\begin{aligned} G_F^{\text{reg}}(x - x') &= \sum_{i=0}^{n} c_i G_F(x - x', M_i)\,, \\ c_0 = 1\,, &\qquad M_0 = m\,, \end{aligned} \tag{3.7}$$

supplemented by the conditions

$$\sum_i c_i = 0\,, \tag{3.8a}$$

$$\sum_i c_i M_i^2 = 0\,. \tag{3.8b}$$

The first condition (3.8a) cancels the strongest singularities of any Feynman graph, and the second one (3.8b) then cancels remaining singularities. This means, that if a given theory is only logarithmically divergent, condition (3.8a) suffices, and only one auxiliary mass $M_1 = M$ with $c_1 = -1$ is necessary.

Let us illustrate how the first condition removes the quadratic singularity in $G(0)$ of ϕ^4 theory:

$$G_F^{\text{reg}}(x-x') = -\int \frac{d^4p}{(2\pi)^4} e^{\mathrm{i}p(x-x')} \left(\frac{1}{p^2-m^2} - \frac{1}{p^2-M^2} \right)$$
$$= -\int \frac{d^4p}{(2\pi)^4} e^{\mathrm{i}p(x-x')} \frac{M^2+m^2}{(p^2-m^2)(p^2-M^2)}\,, \tag{3.9}$$

which for $x = x'$ is only logarithmically singular instead of quadratically. Obviously, all divergences are recovered in the limits $M_i \to \infty$, $i > 0$.

Let us compute an exemplary graph in ϕ^4 theory:

All in all, Pauli-Villars regularization works fine in scalar field theories and also in QED, but it breaks gauge invariance when considering non-Abelian gauge theories such as QCD. Therefore, dimensional regularization is usually the best choice in those models, as it always preserves gauge invariance. We shall review its properties in the following.

Dimensional regularization.

This regularization scheme was introduced in 1971 by G. 't Hooft and M. J. G. Veltman and is based on the observation that divergent Feynman integrals would become convergent when computed in a smaller space dimension. Therefore, computations are done in arbitrary d-dimensional space and the result of the integration can then be analytically continued to real or even complex values of d. The original ultraviolet divergences then manifest themselves as poles at $d = 4$, typically in the form of Gamma functions.

We illustrate this procedure by repeating our previous example in this scheme: Since we are now considering $4-\varepsilon$ dimensional space-time, the canonical dimension of the coupling λ would change as well, unless we introduce a new parameter $\lambda \to \mu^\varepsilon \lambda$. Hence, using Eqn. (A.53a) in Appendix A.3, the one-loop correction to the two-point function in momentum space is given by

$$\frac{1}{2}\mu^\varepsilon \lambda \int \frac{d^{4-\varepsilon}p}{(2\pi)^{4-\varepsilon}} \frac{1}{p^2-m^2} = -\frac{\mathrm{i}\lambda m^2}{32\pi^2} \left(\frac{4\pi\mu^2}{-m^2} \right)^{\varepsilon/2} \Gamma\left(-1+\varepsilon/2\right)$$
$$\approx \frac{\mathrm{i}\lambda m^2}{16\pi^2\varepsilon} + \text{finite}. \tag{3.10}$$

Observe, that the above integral becomes zero in the limit $m \to 0$. This is a typical feature of dimensional regularization.

3.2. Renormalization and renormalization group

Mass renormalization.

The result we have just obtained in (3.10), is essentially what we denoted earlier as $-\mathrm{i}\Sigma$ — seeEqn. (2.47). Defining the "physical" (or renormalized) mass m_{phys} by the pole of the complete propagator

$$G^{(2)}(k) = \frac{\mathrm{i}}{k^2 - m_{\text{phys}}^2}\,, \tag{3.11}$$

gives on comparison

$$m^2_{\text{phys}} = m^2 + \Sigma = m^2 \left(1 - \frac{\lambda}{16\pi^2\varepsilon}\right), \tag{3.12}$$

which to order λ is equivalent to

$$m^2 = m^2_{\text{phys}} \left(1 + \frac{\lambda}{16\pi^2\varepsilon}\right), \tag{3.13}$$

where the renormalized mass is given by $m^2_{\text{phys}} = -\Gamma^{(2)}(0)$ and is taken to be finite (i.e. the bare mass m^2 is infinite and compensates the 1-loop correction).

An alternative point of view which is quite common is to consider counter terms in the Lagrangian and treat those as an interaction giving rise to additional Feynman rules. In the present case such a counter term would read $\delta\mathcal{L}_1 = -\frac{1}{2}\delta m^2\phi^2$ and its effect is that we now have $\Gamma^{(2)}(k) = k^2 - m^2$ since the divergence is cancelled by the additional "interaction". Hence, in this picture m^2 is a finite quantity (in contrast to before). The introduction of this counter term is equivalent to multiplying m by a renormalization factor Z_m.

The coupling λ is renormalized in a similar way by computing the 1-loop correction to the four point function, i.e. $\lambda_{\text{ren.}} = \Gamma^{(4)}(0)$.

Exercise 8 *Compute the 1-loop correction to the four-point Green function, showing that in dimensional regularisation it results to*

$$\Gamma^{(4)}(p_i) = -\mathrm{i}\lambda\mu^{\varepsilon}\left(1 - \frac{3\lambda}{16\pi^2\varepsilon}\right) + \textit{finite}. \tag{3.14}$$

Renormalization of the wave function ϕ is not necessary at 1-loop level in this model, but it appears at two-loop level. We call the according renormalization factors Z_λ and Z_ϕ.

Renormalization group.

We have the following relation between the n-particle vertex function and its renormalized counter part:

$$\Gamma^{(n)}(p_i, m, \lambda) = Z_\phi^{-n/2}(\lambda\mu^{\varepsilon})\Gamma^{(n)}_r(p_i, m_r, \lambda_r, \mu)\,. \tag{3.15}$$

It the follows that

$$\mu\frac{\partial}{\partial\mu}\Gamma^{(n)} = 0\,, \tag{3.16}$$

and hence

$$\left(-n\mu\frac{\partial}{\partial\mu}\ln\sqrt{Z_\phi} + \mu\frac{\partial}{\partial\mu} + \mu\frac{\partial\lambda_r}{\partial\mu}\frac{\partial}{\partial\lambda_r} + \mu\frac{\partial m_r}{\partial\mu}\frac{\partial}{\partial m_r}\right)\Gamma^{(n)}_r = 0\,. \tag{3.17}$$

Defining the quantities

$$\begin{aligned} \gamma(\lambda_r) &= \mu\frac{\partial}{\partial\mu}\ln\sqrt{Z_\phi}\,, & m\gamma_m(\lambda_r) &= \mu\frac{\partial m_r}{\partial\mu}\,, \\ \beta(\lambda_r) &= \mu\frac{\partial\lambda_r}{\partial\mu}\,, \end{aligned} \tag{3.18}$$

we may write this equation as

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(\lambda_r)\frac{\partial}{\partial\lambda_r}-n\gamma(\lambda_r)+m\gamma_m(\lambda_r)\frac{\partial}{\partial m_r}\right)\Gamma_r^{(n)}=0\,. \tag{3.19}$$

It is called the renormalization group equation and expresses the invariance of the renormalized $\Gamma^{(n)}$ under a change of regularization parameter μ.

Let us now derive a similar equation expressing the invariance of $\Gamma^{(n)}$ under a change of scale, i.e. $p\to tp$, $m\to tm$, $\mu\to t\mu$. Since $\Gamma^{(n)}$ has mass dimension $D=4-n+\varepsilon\left(\frac{n}{2}-1\right)$,

$$\begin{aligned}\Gamma^{(n)}(tp,\lambda,m,\mu)&=t^D\Gamma^{(n)}(p,\lambda,m/t,\mu/t)\,,\\ \left(t\frac{\partial}{\partial t}+m\frac{\partial}{\partial m}+\mu\frac{\partial}{\partial\mu}-D\right)\Gamma^{(n)}&=0\,,\end{aligned} \tag{3.20}$$

and using (3.19),

$$\left(-t\frac{\partial}{\partial t}+\beta\frac{\partial}{\partial\lambda}-n\gamma(\lambda)+m\left(\gamma_m(\lambda)-1\right)\frac{\partial}{\partial m}+D\right)\Gamma^{(n)}(tp,\lambda,m,\mu)=0\,, \tag{3.21}$$

where we have omitted the subscripts r. This equation tells us that a change in t may be compensated by a change in m and λ. Hence, we expect the form

$$\Gamma^{(n)}(tp,\lambda,m,\mu)=f(t)\Gamma^{(n)}(p,\lambda(t),m(t),\mu) \tag{3.22}$$

and differentiating this with respect to t reveals that $t\frac{\partial\lambda(t)}{\partial t}=\beta(\lambda)$. Therefore, $\lambda(t)$ is called a running coupling constant. The zeros of the β-function are called fixed points and depending on the quantum theory several of these may be present.

Exercise 9 *Compute the 1-loop β-function for ϕ^4 theory and show that it results to*

$$\beta(\lambda)=\lim_{\varepsilon\to0}\mu\frac{\partial\lambda}{\partial\mu}=\frac{3\lambda^2}{16\pi^2}>0\,. \tag{3.23}$$

A. Supplemental Material

A.1. Dirac formalism

In this section, we will give a short introduction to the quantization-formalism introduced in 1964 by P.A.M. Dirac which enables to handle constrained Hamiltonian systems. To this end we will mainly follow reference [6].

A.1.1. Hamilton systems with constraints

We start with an action for classical mechanics:

$$S = \int L(q_n, \dot{q}_n) dt \,, \tag{A.1}$$

where L is the Lagrangian and the $\dot{q}_n$ denote the time-derivatives of the generalized coordinates q_n. Variation of this action leads to the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) = \frac{\partial L}{\partial q_n} \,, \tag{A.2}$$

and using the chain rule for the left hand side of this equation leads to

$$\ddot{q}_{n'}\left(\frac{\partial^2 L}{\partial \dot{q}_{n'} \partial \dot{q}_n}\right) = \frac{\partial L}{\partial q_n} - \dot{q}_{n'} \frac{\partial^2 L}{\partial q_{n'} \partial \dot{q}_n} \,. \tag{A.3}$$

$\ddot{q}_{n'}$ can only be determined from this equation if the determinant

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}_{n'} \partial \dot{q}_n}\right) \neq 0 \,, \tag{A.4}$$

is unequal zero. In this case, a Legendre transformation leads to the so-called *Hamiltonian*

$$H(q_n, p_n) \equiv p_n \dot{q}_n(q_n, p_n) - L(q_n, \dot{q}_n(q_n, p_n)), \tag{A.5}$$

where the conjugate momenta p_n are defined as

$$p_n \equiv \frac{\partial L}{\partial \dot{q}_n} \,. \tag{A.6}$$

If the p_n do *not* depend on the $\dot{q}_n$, which is exactly the case when the determinant (A.4) vanishes[1], one gets certain relations

$$\phi_m(q, p) = 0, \tag{A.7}$$

[1] With (A.6) one has $\frac{\partial^2 L}{\partial \dot{q}_{n'} \partial \dot{q}_n} = \frac{\partial p_n}{\partial \dot{q}_{n'}}$.

out of (A.6), so-called *primary constraints*. But even if the transformation is singular, one can easily show that H depends only on q_n and p_n: Variation of the right hand side of (A.5) yields

$$\delta H = p_n \delta\dot{q}_n + \dot{q}_n \delta p_n - \frac{\partial L}{\partial q_n}\delta q_n - \frac{\partial L}{\partial \dot{q}_n}\delta\dot{q}_n = \dot{q}_n \delta p_n - \frac{\partial L}{\partial q_n}\delta q_n, \tag{A.8}$$

where the definition (A.6) was used. Hence, from (A.5) follows

$$\left(\dot{q}_n - \frac{\partial H}{\partial p_n}\right)\delta p_n - \left(\frac{\partial L}{\partial q_n} + \frac{\partial H}{\partial q_n}\right)\delta q_n = 0. \tag{A.9}$$

Note that the variations δp_n and δq_n are *not independent* from each other because of the constraints. Since every function G on the phase space, which vanishes on the subspace $\phi_m = 0$, can be written as a linear combination of the constraints ($G = g_m \phi_m$), one concludes that (A.9) must have the form

$$u_m \frac{\partial \phi_m}{\partial p_{m'}}\delta p_{m'} + u_m \frac{\partial \phi_m}{\partial q_{m'}}\delta q_{m'} = 0. \tag{A.10}$$

(A proof can be found in e.g. reference [6].) Comparing coefficients finally leads to the generalized Hamiltonian equations of motion

$$\dot{q}_n = \frac{\partial H}{\partial p_n} + u_m \frac{\partial \phi_m}{\partial p_n}, \tag{A.11a}$$

$$\dot{p}_n = -\frac{\partial H}{\partial q_n} - u_m \frac{\partial \phi_m}{\partial q_n}, \tag{A.11b}$$

where the equations (A.2) and (A.6) were used for the left hand side of (A.11b).

We will now need the definition of the Poisson bracket

$$\{f, g\}_{\text{PB}} = \frac{\partial f}{\partial q_n}\frac{\partial g}{\partial p_n} - \frac{\partial f}{\partial p_n}\frac{\partial g}{\partial q_n}, \tag{A.12}$$

which is antisymmetric and fulfills the Jacobi identity. Let g be an arbitrary function of q_n and p_n. Its time derivative is then given by

$$\dot{g} = \frac{\partial g}{\partial q_n}\dot{q}_n + \frac{\partial g}{\partial p_n}\dot{p}_n . \tag{A.13}$$

Using the Hamiltonian equations of motion (A.11) and the definition of the Poisson bracket (A.12) one obtains

$$\dot{g} = \{g, H\}_{\text{PB}} + u_m \{g, \phi_m\}_{\text{PB}} . \tag{A.14}$$

Following the notation of Dirac we write this expression as a "weak" relation:

$$\dot{g} \approx \{g, H_T\}_{\text{PB}}, \tag{A.15}$$

where H_T denotes the "total" Hamiltonian, $H_T \equiv H + u_m \phi_m$. The "$\approx$" means that one has to evaluate all Poisson brackets before setting the constraints to zero ($\phi_m \approx 0$).

Since the constraints ϕ_m are functions of q_n and p_n as well, they must of course fulfill the same equation of motion (A.14) as the functions g. Hence, consistency demands:

$$0 \approx \dot{\phi}_m \approx \{\phi_m, H\}_{\text{PB}} + u_{m'}\{\phi_m, \phi_{m'}\}_{\text{PB}}\,. \tag{A.16}$$

This equation can be used to determine the u_m unless the second Poisson bracket vanishes ($\{\phi_m, \phi_{m'}\}_{\text{PB}}$=0). In that case one gets further constraints, so-called *secondary* constraints:

$$\chi(q,p) \approx 0\,. \tag{A.17}$$

Of course the secondary constraints have to fulfill the equation of motion (A.14), too. This may lead to further secondary constraints and so on.

Once all secondary constraints have been found, one can start to classify them. According to Dirac there are two "classes" of constraints: *first class* and *second class* (not to be confused with *primary* and *secondary*). A phase space function (or constraint) is called *first class* when its Poisson brackets with all other constraints is (weakly) zero. If at least one of these Poisson brackets is unequal zero, one speaks of a *second class* function/constraint.

Let us go back to equation (A.16): As long as the second Poisson bracket does *not* vanish, one gets solutions for u_m:

$$u_m = U_m(p,q) + v_a V_{am}, \tag{A.18}$$

where U_m are special solutions of the inhomogeneous equation and V_{am} are solutions of the homogeneous equation

$$V_{am}\{\phi_j, \phi_m\}_{\text{PB}} = 0\,. \tag{A.19}$$

The v_a are arbitrary parameters, which means that some kind of freedom is contained in the theory. Consider a dynamic variable $g(t)$ with an initial value $g(0) \equiv g_0$: After the infinitesimal time interval δt one has

$$g(\delta t) = g_0 + \dot{g}\delta t = g_0 + \{g, H_T\}_{\text{PB}}\delta t\,, \tag{A.20}$$

where (A.15) was used. Defining

$$H' \equiv H + U_m\phi_m \quad \text{and} \quad \phi_a \equiv V_{am}\phi_m\,, \tag{A.21}$$

(see (A.18)), one may write

$$g(\delta t) = g_0 + \delta t\left(\{g, H'\}_{\text{PB}} + v_a\{g, \phi_a\}_{\text{PB}}\right)\,. \tag{A.22}$$

Due to (A.19) and the product rule for the Poisson bracket it is obvious that ϕ_a is a first-class constraint[2]. Furthermore, H' is also a first class function by construction (cf. (A.16)). As noticed earlier, v_a are arbitrary parameters, which means that $g(\delta t)$ is ambiguous as well: Replacing v_a with some v'_a in (A.22) leads to a different value for $g(\delta t)$, the deviation being

$$\Delta g(\delta t) = \delta t(v_a - v'_a)\{g, \phi_a\}_{\text{PB}} \equiv \epsilon_a\{g, \phi_a\}_{\text{PB}}\,. \tag{A.23}$$

[2] $\{\phi_j, \phi_a\}_{\text{PB}} = V_{am}\{\phi_j, \phi_m\}_{\text{PB}} + \{\phi_j, V_{am}\}_{\text{PB}}\phi_m \approx 0$.

If one interprets (A.23) as a *gauge transformation*, then the first-class constraints ϕ_a are obviously its generators. In doing two successive gauge transformations of g, one can easily show that the Poisson bracket $\{\phi_a, \phi_{a'}\}_{\text{PB}}$ generates a gauge transformation as well. By applying the product rule one can furthermore show that the Poisson bracket of two first class constraints is a first-class constraint itself. Hence, $\{\phi_a, \phi_{a'}\}_{\text{PB}}$ must be a linear combination of the first-class constraints in the model under consideration. Therefore, we deduce that all primary and secondary first-class constraints generate gauge transformations[3]. This fact should also be taken into account in the equations of motion. We therefore define the *extended Hamiltonian*

$$H_E \equiv H_T + v'_{a'}\phi_{a'} \,. \tag{A.24}$$

The generators $\phi_{a'}$ are all those which are not already contained in H_T and are therefore first-class *secondary* constraints. The corresponding equations of motion are now given by

$$\dot{g} \approx \{g, H_E\}_{\text{PB}} \,. \tag{A.25}$$

What about the second class constraints? In order to treat those we first consider the matrix $C_{AB} = \{\phi_A, \phi_B\}_{\text{PB}}$ where the ϕ_A now denote *all* constraints, and for simplicity we assume the *irreducible* case, i.e. that all $\phi_A \approx 0$ are independent from each other. Obviously, $\det C_{AB} \approx 0$ if there is at least one first class constraint among the ϕ_A. Redefining the constraints as $\phi_A \to a_A{}^B \phi_B$ with an appropriate invertible matrix $a_A{}^B$ one can always find an equivalent description of the constraint surface in terms of constraints $\gamma_a \approx 0$, $\chi_\alpha \approx 0$, whose Poisson bracket matrix reads *weakly*

$$\begin{array}{c} \\ \gamma_b \\ \chi_\beta \end{array} \begin{array}{c} \begin{array}{cc} \gamma_a & \chi_\alpha \end{array} \\ \begin{pmatrix} 0 & 0 \\ 0 & C_{\beta\alpha} \end{pmatrix} \end{array} , \tag{A.26}$$

where $C_{\beta\alpha}$ is an antisymmetric matrix that is everywhere invertible on the constraint surface. In this representation, the constraints are completely split into first and second classes, and the number of second class constraints is obviously even.

A possible way of treating the second class constraints was invented by Dirac in introducing the so-called *Dirac bracket*

$$\{f, g\}_{\text{DB}} \equiv \{f, g\}_{\text{PB}} - \{f, \chi_\alpha\}_{\text{PB}} C^{\alpha\beta} \{\chi_\beta, g\}_{\text{PB}} \,, \tag{A.27}$$

where $C^{\alpha\beta}$ is the inverse of $C_{\alpha\beta}$. Since the extended Hamiltonian is first class, one can easily verify that H_E still generates the correct equations of motion in terms of the Dirac bracket:

$$\dot{g} \approx \{g, H_E\}_{\text{DB}} \,. \tag{A.28}$$

The original Poisson bracket can be discarded after having served its purpose of distinguishing between first-class and second-class constraints and all the equations of the theory can now be formulated in terms of the Dirac bracket (see ref. [6] for a detailed proof).

[3] Initially, $\phi_a = V_{am}\phi_m$ consisted only of *primary* constraints.

A.1.2. Field theoretic extension

We are now interested in the field theoretic extension of the formalism developed above and illustrate this with an example: free Maxwell theory. The action is given by

$$S = -\frac{1}{4}\int dt \int d^3x\, F_{\mu\nu}F^{\mu\nu}\,, \tag{A.29}$$

with the electromagnetic field tensor $F_{\mu\nu}$ defined in Eqn. (1.10). The fields $A_\mu(t,\vec{x})$ correspond to the $q_n(t)$ in the previous section. The variable $\vec{x}$ can be interpreted as a "continuous" index. According to (A.6) with $\dot{A}_\mu \equiv \partial_0 A_\mu = \frac{\partial A_\mu}{\partial t}$, the conjugate momenta are given by

$$\pi^\mu(\vec{x}) = \frac{\delta}{\delta \dot{A}_\mu(\vec{x})}\left(-\frac{1}{4}\int d^3x' F_{\rho\sigma}(\vec{x}')F^{\rho\sigma}(\vec{x}')\right) = F^{\mu 0}(\vec{x})\,. \tag{A.30}$$

In analogy to $\{q_n, p_{n'}\}_{\mathrm{PB}} = \delta_{nn'}$ we now have

$$\{A_\mu(\vec{x}), \pi^\nu(\vec{x}')\}_{\mathrm{PB}} = \delta_\mu^\nu \delta^3\left(\vec{x}-\vec{x}'\right)\,. \tag{A.31}$$

Due to antisymmetry of the field tensor, equation (A.30) yields the primary constraint

$$\pi^0(\vec{x}) \approx 0\,. \tag{A.32}$$

A Legendre transformation, as defined in (A.5), gives us the Hamiltonian of Maxwell theory

$$H = \int d^3x \left(\pi^\mu \dot{A}_\mu + \frac{1}{4}F^{rs}F_{rs} + \frac{1}{2}F^{r0}F_{r0}\right)\,, \tag{A.33}$$

where Latin indices run from 1 to 3. The constraint (A.32), partial integration and the fact that $F_{r0} = -\pi^r$ yields

$$H = \int d^3x \left(\frac{1}{4}F^{rs}F_{rs} + \frac{1}{2}\pi^r\pi^r - A_0\partial_r\pi^r\right)\,. \tag{A.34}$$

All time derivatives have now been replaced by conjugate momenta enabling us to use the consistency condition (A.16) to get

$$0 \approx \dot{\pi}^0 \approx \{\pi^0, H\}_{\mathrm{PB}} = \partial_r\pi^r,$$

which yields the secondary constraint

$$\partial_r\pi^r \approx 0. \tag{A.35}$$

A further consistency check shows that (A.32) and (A.35) are the only constraints, since $\{\partial_r\pi^r, H\}_{\mathrm{PB}} = 0$. Furthermore, they are *first-class* because of

$$\{\pi^0(\vec{x}), \pi^0(\vec{x}')\}_{\mathrm{PB}} = 0\,, \qquad \{\pi^0(\vec{x}), \partial_r\pi^r(\vec{x}')\}_{\mathrm{PB}} = 0\,,$$
$$\{\partial_r\pi^r(\vec{x}), \partial_r\pi^r(\vec{x}')\}_{\mathrm{PB}} = 0\,.$$

Obviously, the Hamiltonian H is first-class as well and therefore can be used for H' from (A.21). The total Hamiltonian H_T hence becomes

$$H_T = \int \left(\frac{1}{4}F^{rs}F_{rs} + \frac{1}{2}\pi^r\pi^r\right)d^3x - \int A_0\partial_r\pi^r d^3x + \int v(\vec{x})\pi^0 d^3x\,, \tag{A.36}$$

where $v(\vec{x})$ is arbitrary. Inserting A_0 into the equation of motion (A.15), we see that $v(\vec{x}) = \dot{A}_0(\vec{x})$. This means that the time derivative of A_0 is *ambiguous* and that A_0 as well as its conjugate momentum $\pi^0 = 0$ are *unphysical*. Using the "extended" Hamiltonian H_E one may eliminate these unphysical quantities:

$$H_E = H_T + \int u(x)\partial_r \pi^r d^3x \,. \tag{A.37}$$

Choosing $v(x) = 0$ and $u'(x) = u(x) - A_0$ one arrives at the new (simplified) Hamiltonian (cf. (A.36))

$$H = \int \left(\frac{1}{4}F^{rs}F_{rs} + \frac{1}{2}\pi^r\pi^r\right) d^3x + \int u'(x)\partial_r\pi^r d^3x \,, \tag{A.38}$$

which still produces the correct equations of motion for all physically relevant variables.

A.2. Natural Units

In general it is always convenient to use a unit system where the formulae under consideration are rendered particularly simple, i.e. with only a minimal set of constants. In particle physics this is achieved by setting the velocity of light c and Planck's constant $\hbar$ equal to 1 and dimensionless:

$$c = \hbar = 1 \,. \tag{A.39}$$

Quantities computed in these units can always be converted to SI units at a later point, if desired. However, let us take a closer look at the properties of these so-called *natural units*. From the speed of light being now dimensionless it follows immediately that

$$1 = [c] = \frac{[\text{length}]}{[\text{time}]} \quad \Rightarrow \quad [\text{length}] = [\text{time}] \,. \tag{A.40}$$

Time and length now have the same dimension. This also becomes clear when considering the relativistic formulation of space-time where the four-vector $x^\mu = (ct, \vec{x})$ with $c = 1$.

Similarly, one finds:

$$1 = [\hbar] = [\text{energy}][\text{time}] \quad \Rightarrow \quad [\text{time}] = \frac{1}{[\text{energy}]} \,. \tag{A.41}$$

So time (and hence length) now has dimension of energy^{-1}. In particle physics one obviously has to deal with the energy of particles, which is to small for the unit Joule to be practical. Therefore, one uses the unit of *electron volts* instead, where

$$1eV = 1,602177 \cdot 10^{-19} J \,, \tag{A.42}$$

which corresponds to the kinetic energy an unbounded electron gets when accelerated by an electrostatic potential of 1 Volt (Volt = Joule/Coulomb, cp. charge of an electron from Table A.1).

From $E = mc^2$ follows furthermore that mass now has dimension of energy as well, i.e.

$$[\text{mass}] = [\text{energy}], \tag{A.43}$$

and if we consider Boltzmann's constant $k_B = 1$, temperature also has the dimension of energy ([temperature] = [energy]). In the same way, $\mu_0 = 1$ and dimensionless leads to voltage and current having the same units (cp. Table A.1). From the relation $\epsilon_0 = \frac{1}{c^2 \mu_0}$, it follows additionally, that the dielectric constant in vacuum $\epsilon_0 = 1$ and dimensionless as well in natural units.

Typical values are:

$10^{-2} eV$:	thermal energy of a particle at room temperature
some eV:	chemical reactions
$5,11 \cdot 10^5 eV = 511 keV$:	rest mass of an electron
some MeV (i.e. $10^6 eV$):	nuclear reactions
$938 MeV = 0,938 GeV$:	rest mass of a proton

speed of light:	$c = 299792458 \frac{m}{s}$
Planck's constant:	$h = 6,62607 \cdot 10^{-34} Js$
	$\hbar = \frac{h}{2\pi} = 1,05457 \cdot 10^{-34} Js$
unit charge:	$e = 1,602177 \cdot 10^{-19} C$ (electron: $q = -e$)
gravitational constant:	$G = 6.673 \cdot 10^{-11} \frac{m^3}{kg\, s^2}$
Boltzmann's constant:	$k_B = 1,3807 \cdot 10^{-23} \frac{J}{K}$
magnetic constant:	$\mu_0 = 4\pi \cdot 10^{-7} \frac{Vs}{Am}$

Table A.1.: some constants of nature

A.3. Dimensional regularization

We follow the derivation given in reference [3] on pages 382-385: Working in n-dimensional Minkowski space we start with integrals of the type

$$I_n(q) = \int \frac{d^n k}{(k^2 + 2kq - L^2)^\alpha}\,, \tag{A.44}$$

where $k = (k_0, r, \phi, \theta_1, \theta_2, \ldots, \theta_{n-3})$ in polar coordinates and the volume element is therefore given by

$$d^n k = dk_0 r^{n-2} dr d\phi \prod_{i=1}^{n-3} \sin^i \theta_i d\theta_i\,. \tag{A.45}$$

Shifting variables $k'_\mu = k_\mu + q_\mu$ and using

$$\int_0^{2\pi} d\phi \prod_{i=1}^{n-3} \int_0^{\pi} \sin^i \theta_i d\theta_i = \frac{2\pi^{\frac{(n-1)}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}\,, \tag{A.46}$$

we arrive at

$$I_n(q) = \frac{2\pi^{\frac{(n-1)}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \int_{-\infty}^{\infty} dk_0 \int_0^{\infty} \frac{r^{n-2} dr}{\left(k_0^2 - r^2 - (q^2 + L^2)\right)^\alpha}\,, \tag{A.47}$$

where the primes have been dropped. Since the integrand depends only quadratically on k_0 we can replace $\int_{-\infty}^{\infty} dk_0 \to 2\int_0^{\infty} dk_0$. To evaluate the remaining integrals we use the Euler beta function

$$\mathcal{B}(x,y) \equiv \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = 2\int_0^{\infty} dt t^{2x-1}(1+t^2)^{-x-y}\,, \qquad \begin{matrix}\Re(x)>0\\ \Re(y)>0\end{matrix} \tag{A.48}$$

Considering

$$x = \frac{1+\beta}{2}, \qquad y = \alpha - \frac{1+\beta}{2}, \qquad t = \frac{s}{M}\,, \tag{A.49}$$

leads to

$$\int_0^{\infty} ds \frac{s^{\beta}}{(s^2+M^2)^{\alpha}} = \frac{\Gamma\left(\frac{1+\beta}{2}\right)\Gamma\left(\alpha-\frac{1+\beta}{2}\right)}{2\,(M^2)^{\alpha-(1+\beta)/2}\,\Gamma(\alpha)}\,, \tag{A.50}$$

and if we identify $\beta = 0$ and $M^2 = -r^2 - (q^2+L^2)$ we can use this formula to perform the integration over k_0 in (A.47):

$$I_n(q) = \frac{2\pi^{\frac{(n-1)}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}\int_0^{\infty} dr \frac{r^{n-2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha-\frac{1}{2}\right)}{(-r^2-(q^2+L^2))^{\alpha-1/2}\,\Gamma(\alpha)}\,. \tag{A.51}$$

By identifying $\alpha' = \alpha - \frac{1}{2}$, $\beta = n-2$ and $M^2 = q^2+L^2$ we can once again use formula (A.50) to perform the remaining integral:

$$I_n(q) = (-1)^{\frac{1}{2}-\alpha}\frac{\pi^{\frac{n}{2}}}{\Gamma(\alpha)}\frac{\Gamma\left(\alpha-\frac{n}{2}\right)}{(q^2+L^2)^{\alpha-\frac{n}{2}}} = (-1)^{\frac{n}{2}} i\pi^{\frac{n}{2}}\frac{\Gamma\left(\alpha-\frac{n}{2}\right)}{\Gamma(\alpha)}(-q^2-L^2)^{\frac{n}{2}-\alpha}\,. \tag{A.52}$$

Therefore the result is

$$I_n(q) = \int \frac{d^n k}{(k^2+2kq-L^2)^{\alpha}} = (-1)^{\frac{n}{2}} i\pi^{\frac{n}{2}}\frac{\Gamma\left(\alpha-\frac{n}{2}\right)}{\Gamma(\alpha)}(-q^2-L^2)^{\frac{n}{2}-\alpha}\,. \tag{A.53a}$$

Differentiating both sides with respect to q_μ and redefining α as well as using the property of the Gamma-function $x\Gamma(x) = \Gamma(1+x)$ leads to

$$I_n^{\mu}(q) = \int d^n k \frac{k^{\mu}}{(k^2+2kq-L^2)^{\alpha}} = (-q^{\mu})I_n(q)\,, \tag{A.53b}$$

and further differentiation with respect to q_ν yields

$$I_n^{\mu\nu}(q) = \int d^n k \frac{k^{\mu}k^{\nu}}{(k^2+2kq-L^2)^{\alpha}} = \left(q^{\mu}q^{\nu} + \frac{g^{\mu\nu}\left(-q^2-L^2\right)}{2\alpha-n-2}\right) I_n(q)\,. \tag{A.53c}$$

A useful trick in order to bring integrals appearing in typical Feynman graphs into the form $I_n(0)$ or $I_n^{\mu\nu}(0)$ as defined above, is given by Feynman's formula

$$\frac{1}{ab} = \int_0^1 \frac{dz}{(az+b(1-z))^2}\,. \tag{A.54}$$

The proof of this integral formula is straightforward: simply substitute $z' = (a-b)z - b$.

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